A TREATISE

ON

DIFFERENTIAL EQUATIONS,

AND ON THE

CALCULUS OF FINITE DIFFERENCES.

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37,	16,	$\int dx \frac{dp}{dx}$	$\int dx\, x \frac{dp}{dx}$.
38,	5,	$\frac{d}{(1+p^2)^{\frac{1}{2}}}$	$\frac{a}{(1+p^2)^{\frac{1}{2}}}$.
69,	2,	Ex. 5	Ex. 6.
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49,	8,	Art. 72	Art. 67.
98,	20,	$\cos_\theta$	$\cos\theta$.

INTEGRATION

OF

DIFFERENTIAL EQUATIONS

BETWEEN TWO OR MORE VARIABLES.

SECTION I.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE.

1. In that part of the Integral Calculus which relates to the integration of explicit functions of one variable, we have to determine the relation between y and x from the equation

$$\frac{dy}{dx} = f(x);$$

in the present portion, we have to determine it from the equation

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \dots \frac{dy}{dx}, y, x\right) = 0;$$

or to assign the relation between x, y, z (where z is a function of the independent variables x and y), or between a greater number of variables and their functions, from the equation

$$f\left(\frac{dz}{dx}, \frac{dz}{dy}, z, x, y\right) = 0,$$

or from other equations in which a greater number of variables and differential coefficients of higher orders are involved.

2. A differential equation is said to be of the n^{th} order, when the differential coefficient of the highest order which it involves is the n^{th}.

A differential equation of any order is said, moreover, to be of the first, second, &c., degree, when the differential coefficient

which marks its order, is raised to the first, second, &c., power: or when it involves a product at most of m dimensions in differential coefficients and their powers, it is said to be of the m^{th} degree.

To integrate a differential equation of any order, is to pass to the primitive equation between the variables and the constants, from which the proposed may have been derived by the process of differentiation.

3. We shall begin with the simplest case, viz. that of differential equations of the first order and degree, which will be of the form

$$M + N\frac{dy}{dx} = 0,$$

M and N being functions of x and y.

Every differential equation of the first order and degree is either the direct derived equation of a primitive; or it results from the combination of the derived equation with its primitive, so as to eliminate a constant which enters in each only to the first power; the former sort are called exact, the latter inexact.

4. First, let $u = f(x, y) = 0$ be an equation between x and y, by virtue of which y is a function of x; then, as is proved in the Differential Calculus, $\frac{dy}{dx}$ is given by the equation

$$\frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0,$$

the partial differential coefficients $\frac{du}{dx}$, $\frac{du}{dy}$, being formed as if the variables x and y were independent of one another; or, since $\frac{du}{dx}$, $\frac{du}{dy}$, are functions of x and y which we may represent by M and N,

$$M + N\frac{dy}{dx} = 0,$$

a differential equation of the first order and degree, of which $f(x, y) = 0$ is the primitive or integral. Now if $f(x, y)$,

besides other constants which are affected with x and y, contain a term $+C$ independent of x and y, this will not enter into M and N, having disappeared in differentiating; and if there be no such term, we may add it, and $f(x, y) + C=0$ is still a relation between x and y which satisfies the equation

$$M + N\frac{dy}{dx} = 0;$$

under this form it is called the complete integral; and the constant C, which does not appear in the differential equation, is called the arbitrary constant; if the integral did not contain such a term as $+C$, it would not be sufficiently general, and would be only a particular case of the complete integral.

We shall presently give the test which every equation of this sort must satisfy, and the mode of integrating it. It is evident that no equation of the first order which is not of the first degree can be exact.

5. Next, let C_1 be another constant which enters to the first power in the equation

$$f(x, y) + C = 0,$$

then C_1 will be affected with x and y, and will consequently appear to the first power in

$$M + N\frac{dy}{dx} = 0;$$

and if a value of C_1 be obtained from either of these equations and substituted in the other, the result will be

$$M_1 + N_1\frac{dy}{dx} = 0,$$

an equation of the first order and degree, involving all the constants which enter into $f(x, y) + C = 0$, except C_1. Hence whilst the direct derived equation of

$$f(x, y) + C = 0, \text{ viz. } M + N\frac{dy}{dx} = 0,$$

does not involve the term C which is independent of x and y, there will be as many other differential equations of the first

order and degree that have $f(x, y) + C = 0$ for their primitive, as it has independent constants entering only in the first power; if any constant enter in a dimension above the first, the differential equation obtained by eliminating it, will evidently not be of the first degree.

There are two principal methods of integrating equations of this sort, which consist either in separating the variables, by substitution, or some algebraical process; or in restoring the factor which makes them exact.

Exact Differential Equations of the First Order.

6. Let y be a function of x determined by the equation

$$u = f(x, y) = 0;$$

then the equation which gives the value of $\frac{dy}{dx}$ is

$$\frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0, \quad \text{or} \quad M + N\frac{dy}{dx} = 0;$$

the differential coefficients $\frac{du}{dx}, \frac{du}{dy}$ being formed on the hypothesis that the variables x and y are independent of one another; then, as proved in the Differential Calculus,

$$\frac{dM}{dy} = \frac{dN}{dx}.$$

Conversely, an equation of the form $M + N\frac{dy}{dx} = 0$ being proposed in which M and N are functions of x and y, if the condition

$$\frac{dM}{dy} = \frac{dN}{dx}$$

(which is called the criterion of integrability) be satisfied, the equation results from the immediate differentiation of an equation of the form $f(x, y) = 0$; and to find its integral amounts to finding a function of two variables $f(x, y)$ whose differential shall be $Mdx + Ndy$, and then to put $f(x, y)$ equal to a constant; if the above condition be not satisfied, there exists no

equation by the simple differentiation of which, the given equation can be produced.

7. To integrate the exact differential equation

$$M+N\frac{dy}{dx}=0.$$

Let the equation from which it is derived be

$$u=f(x, y)=0;$$

$$\text{then } \frac{du}{dx}=M, \quad \frac{du}{dy}=N, \quad \text{and } \frac{dM}{dy}=\frac{dN}{dx};$$

$$\therefore\ u=\int dxM+Y,$$

denoting by Y a function of y which may have disappeared, since M is the differential coefficient of u relative to x, on the hypothesis that x and y are independent;

$$\therefore\ \frac{du}{dy}=\frac{d}{dy}(\int dxM)+\frac{dY}{dy}=N,$$

$$\therefore\ \frac{dY}{dy}=N-\frac{d}{dy}(\int dxM), \quad \text{and } Y=\int dy\,\{N-\frac{d}{dy}(\int dxM)\},$$

$$\therefore\ u=\int dxM+\int dy\,\{N-\frac{d}{dy}(\int dxM)\}+C=0,$$

the complete integral involving one arbitrary constant.

8. Obs. The equation $Y=\int dy\,\{N-\frac{d}{dy}(\int dxM)\}$ will be absurd, unless the expression $N-\frac{d}{dy}(\int dxM)$ be independent of x: therefore its differential coefficient with respect to x must vanish;

$$\therefore\ \frac{dN}{dx}-\frac{d}{dx}\frac{d}{dy}(\int dxM)=\frac{dN}{dx}-\frac{d}{dy}\frac{d}{dx}(\int dxM)=\frac{dN}{dx}-\frac{dM}{dy}$$

must equal zero; which it does, since the criterion of integrability is supposed to be satisfied. Hence it will be necessary only to integrate those terms in N which involve y only; and if N can be reduced to such a form as to contain no such terms, the solution will be $u=\int dxM+C=0$.

9. As the simplest case of exact equations, we may first notice those in which the variables are separated; they will be of the form

$$X + Y\frac{dy}{dx} = 0,$$

where X denotes a function of x only, and Y a function of y only; here the criterion of integrability is manifestly satisfied, for

$$\frac{dX}{dy} = \frac{dY}{dx} = 0;$$

and the complete integral is

$$\int dx X + \int dx Y\frac{dy}{dx} = C, \quad \text{or } \int dx X + \int dy Y = C.$$

To this case may likewise be reduced the equation

$$XY_1 + YX_1\frac{dy}{dx} = 0,$$

which becomes, when divided by $X_1 Y_1$,

$$\frac{X}{X_1} + \frac{Y}{Y_1}\frac{dy}{dx} = 0.$$

Ex. 1. $$\frac{1}{\sqrt{a^2 - x^2}} + \frac{1}{\sqrt{a^2 - y^2}}\frac{dy}{dx} = 0;$$

$$\therefore \sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{a} = \sin^{-1}\frac{C}{a},$$

$$\text{or } \sin^{-1}\left(\frac{x}{a}\sqrt{1 - \frac{y^2}{a^2}} + \frac{y}{a}\sqrt{1 - \frac{x^2}{a^2}}\right) = \sin^{-1}\frac{C}{a},$$

$$\text{or } x\sqrt{a^2 - y^2} + y\sqrt{a^2 - x^2} = aC;$$

at which we may also arrive, by multiplying the proposed equation by xy, and integrating by parts, which gives

$$-y\sqrt{a^2 - x^2} + \int dx \sqrt{a^2 - x^2}\frac{dy}{dx} - x\sqrt{a^2 - y^2} + \int dx \sqrt{a^2 - y^2} + C = 0,$$

or, since the part affected by the sign $\int dx$ viz.

$$\sqrt{a^2-y^2}+\sqrt{a^2-x^2}\frac{dy}{dx},$$

is equal to zero by the proposed,

$$y\sqrt{a^2-x^2}+x\sqrt{a^2-y^2}=C.$$

Ex. 2. $1+y+y^2+(1+x+x^2)\dfrac{dy}{dx}=0,$

$$C(x+y+1)=2xy+x+y-1.$$

10. The following are instances of the integration of exact differential equations by the method of Art. 7.

Ex. 1. $ax+by+c+(bx+my+n)\dfrac{dy}{dx}=0.$

$$\therefore \frac{dM}{dy}=b=\frac{dN}{dx},$$

$$\frac{du}{dx}=ax+by+c; \quad \therefore u=\frac{ax^2}{2}+(by+c)x+Y,$$

$$\frac{du}{dy}=bx+\frac{dY}{dy}=bx+my+n;$$

$$\therefore \frac{dY}{dy}=my+n; \quad \therefore Y=\tfrac{1}{2}my^2+ny+C;$$

$$\therefore \tfrac{1}{2}(ax^2+my^2)+(by+c)x+ny+C=0.$$

Ex. 2. $\dfrac{1}{\sqrt{x^2+y^2}}+\left(\dfrac{1}{y}-\dfrac{x}{y\sqrt{x^2+y^2}}\right)\dfrac{dy}{dx}+c=0,$

$$\log(x+\sqrt{x^2+y^2})+cx+C=0.$$

Ex. 3. $\dfrac{2y-2x\dfrac{dy}{dx}}{y\sqrt{x^2-y^2}}+c=0,\quad \log\left(\dfrac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}}\right)+cx+C=0.$

Ex. 4. $\dfrac{y^2+ax+(1-xy)\dfrac{dy}{dx}}{y^3+3ayx-a+a^2x^3}=0.$ Shew this to be exact.

Ex. 5. $(x^2+y^2-a^2)\dfrac{dy}{dx}+(x^2+2xy+a^2)=0,$

$$a^2x+x^2y+\tfrac{1}{3}x^3+\tfrac{1}{3}y^3-a^2y=C.$$

Ex. 6. $\dfrac{x^4\dfrac{dy}{dx}+x^4y^2-a}{x^4(1-xy)^2-ax^2}=0$, here $\int dxM+C=0$,

$$\text{or } \log\frac{\sqrt{a}+x(1-xy)}{\sqrt{a}-x(1-xy)}-\frac{2\sqrt{a}}{x}+C=0.$$

Homogeneous Equations.

11. We come next to the case of inexact equations, in which the variables are separable by substitution; of these the most important class is homogeneous equations.

Let $M+N\dfrac{dy}{dx}=0$ be a homogeneous equation, that is, one in which each of the functions M and N is or can be expressed by series of the form

$$M=ay^mx^{r-m}+by^nx^{r-n}+cy^px^{r-p}+\&c.,$$
$$N=\alpha y^\mu x^{r-\mu}+\beta y^\nu x^{r-\nu}+\gamma y^\pi x^{r-\pi}+\&c.,$$

the sum of the dimensions of x and y in each term of M and N being equal to r.

Let $y=xz$ where z denotes a new function of x, then

$$M=x^r\left(a\frac{y^m}{x^m}+b\frac{y^n}{x^n}+\&c.\right)=x^rf(z),$$

$$N=x^r\left(\alpha\frac{y^\mu}{x^\mu}+\beta\frac{y^\nu}{x^\nu}+\&c.\right)=x^r\phi(z),$$

$$\frac{dy}{dx}=z+x\frac{dz}{dx}.$$

Hence, making these substitutions in the given equation, and dividing by x^r,

$$f(z)+\phi(z)\left(z+x\frac{dz}{dx}\right)=0,\quad\text{or } \frac{1}{x}+\frac{1}{\dfrac{f(z)}{\phi(z)}+z}\cdot\frac{dz}{dx}=0,$$

in which the variables are separated. Similarly, the variables may be separated by making $x=yz$; and the latter substitution

will be more convenient when N is a more complicated expression than M.

Hence it is easy to effect the separation of the variables in equations which are either homogeneous, or can be made homogeneous; besides these, the number of equations in which that separation is possible, is very limited.

Ex. 1. $3y^2x + 2x^3 + y^3\dfrac{dy}{dx} = 0.$

$$\text{Here } \frac{M}{N} = \frac{3y^2x + 2x^3}{y^3} = 3\frac{x}{y} + 2\frac{x^3}{y^3} = \frac{3}{z} + \frac{2}{z^3};$$

$$\therefore \frac{1}{x} + \frac{1}{\dfrac{3}{z} + \dfrac{2}{z^3} + z} \cdot \frac{dz}{dx} = 0, \quad \text{or } \frac{1}{x} + \frac{z^3}{z^4 + 3z^2 + 2}\frac{dz}{dx} = 0,$$

$$\text{or } \frac{1}{x} + \left(\frac{2z}{z^2+2} - \frac{z}{z^2+1}\right)\frac{dz}{dx} = 0;$$

$$\therefore \log x + \log(z^2 + 2) - \tfrac{1}{2}\log(z^2+1) = \log C;$$

$$\therefore \frac{x(z^2+2)}{\sqrt{z^2+1}} = C; \quad \text{or } y^2 + 2x^2 = C\sqrt{x^2+y^2}.$$

2. $y^2 + (xy + x^2)\dfrac{dy}{dx} = 0, \qquad y = C\sqrt{1 + 2\dfrac{y}{x}}.$

3. $(x - y)\dfrac{dy}{dx} = x + y, \qquad \tan^{-1}\dfrac{y}{x} = \log\sqrt{\dfrac{x^2+y^2}{C^2}}.$

4. $(x^2 - y^2)\dfrac{dy}{dx} - 2xy = 0, \quad x^2 + y^2 = Cy.$

5. $x\dfrac{dy}{dx} - y = \sqrt{x^2 - y^2}, \qquad \sin^{-1}\dfrac{y}{x} = \log\dfrac{x}{C}.$

6. $x\dfrac{dx}{dy} + y = \sqrt{x^2 + y^2}, \qquad 2Cy + C^2 = x^2.$

7. $\sqrt{y} + (\sqrt{y} - \sqrt{x})\dfrac{dy}{dx} = 0,$

$$\log(y - \sqrt{xy} + x) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt{x} - \sqrt{y}}{\sqrt{3y}}\right) = C.$$

12. In the following instances, the equations are not homogeneous, but are made so by simple substitutions.

Ex. 1. $ax + by + c + (mx + ny + p)\dfrac{dy}{dx} = 0.$

Let $ax + by + c = z, \quad mx + ny + p = v,$

z and v denoting functions of x; then $\dfrac{dy}{dx} = -\dfrac{z}{v}$, and

$$a + b\frac{dy}{dx} = \frac{dz}{dx}, \quad m + n\frac{dy}{dx} = \frac{dv}{dx};$$

$$\therefore \frac{mv - nz}{av - bz} = \frac{dv}{dz} \text{ or } mv - nz + (bz - av)\frac{dv}{dz} = 0;$$

this being homogeneous, assume $v = zw$, w being a function of z,

$$\therefore \frac{1}{z} + \frac{(aw - b)}{n - (m + b)w + aw^2}\frac{dw}{dz} = 0,$$

in which the variables are separated.

Ex. 2. $x^m\left(ay + bx\dfrac{dy}{dx}\right) = y^n\left(\alpha y + \beta x\dfrac{dy}{dx}\right),$

$$\text{or } \frac{1}{y}(bx^m - \beta y^n)\frac{dy}{dx} = (\alpha y^n - ax^m)\frac{1}{x}.$$

Let $x^m = z$, $y^n = v$, z and v being functions of x,

$$\therefore \frac{m}{x} = \frac{1}{z}\frac{dz}{dx}, \quad \frac{n}{y}\frac{dy}{dx} = \frac{1}{v}\frac{dv}{dx}, \quad \therefore \frac{nx}{my}\frac{dy}{dx} = \frac{z}{v}\frac{dv}{dz};$$

$$\therefore (bz - \beta v)\frac{mz}{nv}\frac{dv}{dz} = \alpha v - az,$$

which is homogeneous.

Ex. 3. $\dfrac{dy}{dx} + ay^n x^p + by^q x^m = 0$ will become homogeneous by

making $y = z^{\frac{p+1}{1-n}}$, the equation of condition between

m, n, p, q, being $(p + 1)(1 - q) = (m + 1)(1 - n).$

Ex. 4. $\dfrac{dy}{dx} = \dfrac{xy^3}{a^2 + xy}$.

This may be written $-\dfrac{d}{dx}\left(\dfrac{1}{y}\right) = \dfrac{x}{\dfrac{a^2}{y} + x}$,

and therefore becomes homogeneous when z is written for $\dfrac{1}{y}$.

Linear Equations of the First Order.

13. The next important class of inexact equations of the first order which admit of being integrated, are linear equations, the general form of which is

$$\frac{dy}{dx} + Py = Q,$$

P and Q being functions of x; they are called linear because they involve no power of y above the first.

Assume $y = vz$, v and z being functions of x,

$$\therefore\ v\frac{dz}{dx} + z\frac{dv}{dx} + Pvz = Q.$$

Now z being an indeterminate quantity, may be assumed so that the equation last written down may resolve itself into two others, each of which admits of the separation of its variables; to this end let

$$v\frac{dz}{dx} + Pzv = 0,$$

or, dividing by v, $\dfrac{dz}{dx} + Pz = 0$, or $\dfrac{1}{z}\dfrac{dz}{dx} + P = 0$;

$$\therefore\ \log z = -\int dxP, \text{ or } z = e^{-\int dxP}.$$

The remaining part of the equation gives

$$z\frac{dv}{dx} = Q, \text{ or, substituting for } z,\ \frac{dv}{dx} = Qe^{\int dxP};$$

$$\therefore\ v = \int dx\, Qe^{\int dxP} + C,$$

$$\text{and } y = e^{-\int dxP}\left\{\int dx\, Qe^{\int dxP} + C\right\},$$

the complete primitive involving one arbitrary constant.

Obs. It is unnecessary to add a constant after performing the integration indicated in the equation $z=e^{-\int dxP}$; for let

$$z=e^{-\int dxP+C'}=C_1e^{-\int dxP};$$

then $y=zv=C_1e^{-\int dxP}\left\{\frac{1}{C_1}\int dx\,Qe^{\int dxP}+C\right\}=e^{-\int dxP}\,(\int dx\,Qe^{\int dxP}+CC_1)$,

which is the same result as before, since CC_1 is equivalent only to a single constant.

14. If we differentiate the result

$$ye^{\int dxP}=\int dx\,Qe^{\int dxP}+C,$$

$$\text{we get } e^{\int dxP}\left(\frac{dy}{dx}+Py\right)=e^{\int dxP}\,Q,$$

which shews that if we multiply the proposed equation by $e^{\int dxP}$, each member is separately integrable; and this is the most convenient practical mode of integrating it. When it is once known that the factor which makes the equation integrable is a function of x only, its value may be immediately found; for let it be denoted by X; then

$$X\frac{dy}{dx}+(Py-Q)\,X=0 \text{ is exact},$$

$$\therefore\ \frac{dX}{dx}=\frac{d}{dy}(Py-Q)\,X=PX, \text{ or } \frac{1}{X}\frac{dX}{dx}=P,$$

$$\therefore\ \log X=\int dxP, \text{ or } X=e^{\int dxP}.$$

15. It must be observed that if the second member of the equation $\frac{dy}{dx}+Py=Q$ be multiplied by any power of y, it is still reducible to the standard form of a linear equation of the first order. For suppose the equation to be

$$\frac{dy}{dx}+Py=Qy^n,$$

then dividing both sides by y^n, and multiplying by $-(n-1)$, we get

$$\frac{d}{dx}\left(\frac{1}{y^{n-1}}\right)-\frac{1}{y^{n-1}}(n-1)\,P=-\,Q\,(n-1).$$

Hence the factor which makes both sides integrable is $e^{-(n-1)\int dx P}$, and the result is

$$\frac{1}{y^{n-1}} = e^{(n-1)\int dx P}\{-(n-1)\int dx\, Q e^{-(n-1)\int dx P} + C\}.$$

Ex. 1. $\dfrac{dy}{dx} - \dfrac{x}{1+x^2}y = \dfrac{a}{1+x^2}$,

$$P = -\frac{x}{1+x^2}, \quad \int dx\, P = -\tfrac{1}{2}\log(1+x^2) = \log\frac{1}{\sqrt{1+x^2}},$$

$$\therefore\ e^{\int dx P} = \frac{1}{\sqrt{1+x^2}},$$

$$\therefore\ \frac{y}{\sqrt{1+x^2}} = \int dx \frac{a}{(1+x^2)^{\frac{3}{2}}} + C = \frac{ax}{\sqrt{1+x^2}} + C;$$

$$\therefore\ y = ax + C\sqrt{1+x^2}.$$

Ex. 2. $\dfrac{dy}{dx} + y = xy^3$,

$$\text{or,}\quad \frac{d}{dx}\left(\frac{1}{y^2}\right) - \frac{2}{y^2} = -2x,$$

$$P = -2, \quad \int dx P = -2x, \quad e^{\int dx P} = e^{-2x};$$

$$\therefore\ \frac{e^{-2x}}{y^2} = \int dx e^{-2x}(-2x) = e^{-2x}x - \int dx e^{-2x} = e^{-2x}x + \tfrac{1}{2}e^{-2x} + C;$$

$$\therefore\ \frac{1}{y^2} = x + \tfrac{1}{2} + Ce^{2x}.$$

3. $\dfrac{dy}{dx} - xy = x, \quad y = Ce^{\frac{1}{2}x^2} - 1.$

4. $\dfrac{dy}{dx} + \dfrac{xy}{1-x^2} = x\sqrt{y}, \quad \sqrt{y} = C(1-x^2)^{\frac{1}{4}} - \tfrac{1}{3}(1-x^2).$

5. $\dfrac{dy}{dx} + \dfrac{1-2x}{x^2}y = 1, \qquad y = x^2(1 + Ce^{\frac{1}{x}}).$

6. $\dfrac{dy}{dx} = x^3y^3 - xy, \qquad \dfrac{1}{y^2} = x^2 + 1 + Ce^{x^2}.$

7. $\dfrac{dy}{dx} = a\sin x + by, \qquad y = -a\,.\dfrac{b\sin x + \cos x}{1+b^2} + Ce^{bx}.$

Riccati's Equation.

16. There are certain cases of the equation

$$\frac{dy}{dx} + by^2 = ax^m$$

(called Riccati's Equation, after the Mathematician who first considered it) in which the variables are separable.

First, let $m = 0$, then $\frac{dy}{dx} = a - by^2$, or $\dfrac{\frac{dy}{dx}}{a - by^2} = 1$, where the variables are separated.

Secondly, let $m = -2$, and assume $y = \frac{u}{x}$, u being a function of x:

$$\therefore \frac{1}{x}\frac{du}{dx} - \frac{u}{x^2} + \frac{bu^2}{x^2} = \frac{a}{x^2};$$

$$\therefore x\frac{du}{dx} = a + u - bu^2,$$

where the variables are separated.

Thirdly, let $m = -4$, and assume $y = \frac{1}{bx} + \frac{u}{x^2}$,

$$\text{then } -\frac{1}{bx^2} - \frac{2u}{x^3} + \frac{1}{x^2}\frac{du}{dx} + \frac{1}{bx^2} + \frac{2u}{x^3} + \frac{bu^2}{x^4} = \frac{a}{x^4};$$

$$\therefore x^2\frac{du}{dx} + bu^2 = a,$$

where the variables are separated.

17. Besides the above, the variables are likewise separable in the cases when $m = \frac{-4i}{2i \pm 1}$, i being any integer from 0 to infinity; all which values of m evidently lie between 0 and -2.

First, let $m = \frac{-4i}{2i-1}$; assume $y = \frac{1}{bx} + \frac{1}{x^2u}$;

$$\therefore\ by^2 - ax^m = \frac{1}{bx^2} + \frac{2}{x^3u} + \frac{b}{x^4u^2} - ax^m,$$

$$\frac{dy}{dx} = -\frac{1}{bx^2} - \frac{2}{x^3u} - \frac{1}{x^2u^2}\frac{du}{dx};$$

therefore adding these together,

$$\frac{b}{x^4u^2} - ax^m - \frac{1}{x^2u^2}\frac{du}{dx} = 0,$$

$$\text{or } x^2\frac{du}{dx} + au^2x^{m+4} - b = 0.$$

Now, let $x = z^{\frac{1}{m+3}}$, then

$$\frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx} = (m+3)\,z^{\frac{m+2}{m+3}}\frac{du}{dz};$$

$$\therefore\ (m+3)\,z^{\frac{m+4}{m+3}}\frac{du}{dz} + au^2z^{\frac{m+4}{m+3}} - b = 0;$$

$$\therefore\ \frac{du}{dz} + \frac{a}{m+3}u^2 = \frac{b}{m+3}z^{-\frac{m+4}{m+3}}, \text{ or } \frac{du}{dz} + b_1u^2 = a_1z^{m_1}.$$

But $m = \dfrac{-4i}{2i-1}$, $\therefore\ -\dfrac{m+4}{m+3} = -\dfrac{4i-4}{2i-3} = -\dfrac{4(i-1)}{2(i-1)-1} = m_1$.

Hence by these substitutions the equation is transformed into another of exactly the same form, with $i-1$ instead of i in the index of the variable in the second member.

Similarly, by substituting $\dfrac{1}{b_1z} + \dfrac{1}{u_1z^2}$ for u, and $z_1^{\frac{1}{m_1+3}}$ for z, we shall transform the equation into another of the same form where $m_2 = \dfrac{-4(i-2)}{2(i-2)-1}$; and consequently, after i substitutions the index of the variable in the second member will become zero, and the variables will be separated.

Secondly, let $m = \dfrac{-4i}{2i+1}$; assume $y = \dfrac{1}{u}$, then

$$-\frac{1}{u^2}\frac{du}{dx} + \frac{b}{u^2} = ax^m, \text{ or } -\frac{du}{dx} + b = ax^mu^2,$$

let $x = z^{\frac{1}{m+1}}$, then $\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx} = (m+1)\, z^{\frac{m}{m+1}} \frac{du}{dz}$;

$$\therefore\ -(m+1)\, z^{\frac{m}{m+1}} \frac{du}{dz} + b = az^{\frac{m}{m+1}} u^2,$$

$$\text{or } \frac{du}{dz} + \frac{a}{m+1} u^2 = \frac{b}{m+1} z^{-\frac{m}{m+1}},$$

$$\text{but } m = \frac{-4i}{2i+1}, \quad \therefore\ -\frac{m}{m+1} = -\frac{-4i}{-2i+1} = \frac{-4i}{2i-1}.$$

Hence by these substitutions this case is reduced to the former; and therefore when $m = \frac{-4i}{2i \pm 1}$, the variables in the equation $\frac{dy}{dx} + by^2 = ax^m$ can be separated. It may be observed that the more general equation, $\frac{dy}{dx} + by^2x^{q-1} = ax^p$, is reducible to this form by putting $x^q = z$.

18. We shall now give some other instances of equations in which the variables are separable by particular substitutions.

Ex. 1. $a\left(x\frac{dy}{dx} - y\right) = \left(x + y\frac{dy}{dx}\right)\sqrt{x^2 + y^2 - a^2}.$

When an equation contains the expressions

$$y\frac{dy}{dx} + x, \quad x\frac{dy}{dx} - y, \quad \sqrt{x^2 + y^2},$$

the introduction of polar co-ordinates will sometimes effect the separation of the variables; that is, to assume

$$x = \rho \cos\theta, \quad y = \rho \sin\theta,$$

ρ being supposed a function of θ, for then

$$y\frac{dy}{d\theta} + x\frac{dx}{d\theta} = \rho\frac{d\rho}{d\theta}, \quad x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = \rho^2.$$

Hence the proposed equation, which considering x and y as functions of θ, may be written

$$a\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right) = \left(x\frac{dx}{d\theta} + y\frac{dy}{d\theta}\right)\sqrt{x^2 + y^2 - a^2},$$

becomes $a\rho^2 = \rho \dfrac{d\rho}{d\theta} \sqrt{\rho^2 - a^2}$, or $a = \dfrac{1}{\rho} \sqrt{\rho^2 - a^2}\, \dfrac{d\rho}{d\theta}$;

$$\therefore\ a\theta = \sqrt{\rho^2 - a^2} - a \sec^{-1} \frac{\rho}{a} + C,$$

$$\text{or } a \tan^{-1} \frac{y}{x} = \sqrt{x^2 + y^2 - a^2} - a \sec^{-1} \frac{\sqrt{x^2+y^2}}{a} + C.$$

Ex. 2. $y - x\dfrac{dy}{dx} + \dfrac{dy}{dx} \sqrt{(ax+by)\dfrac{x}{y}} = 0,$

$$\text{or } y\frac{dx}{dy} - x + \sqrt{(ax+by)\frac{x}{y}} = 0;$$

assume $\dfrac{x}{y} = z$, z being a function of y,

then $y\dfrac{dx}{dy} - x = y^2\dfrac{dz}{dy}$; $\therefore\ y^2\dfrac{dz}{dy} + \sqrt{(az+b)\,zy} = 0,$

where the variables are separated.

Ex. 3. $\dfrac{dy}{dx} = f(mx+ny)$. Let $mx + ny = z$, then

$\dfrac{dz}{dx} = m + n\dfrac{dy}{dx} = m + nf(z)$, where the variables are separated.

Ex. 4. $\dfrac{dy}{dx} + p + qy + ry^2 = 0$, p, q, and r being functions of x. Having given $y = u$ a particular integral, to find the complete integral.

Assume $y = u + z^{-1}$, then

$\dfrac{dz}{dx} - (q + 2ur)\,z - r = 0$, a linear equation;

$$\therefore\ \frac{1}{y-u} = z = e^{\int dx\,(q+2ur)} \times \int dx\, r e^{-\int dx\,(q+2ru)}.$$

Thus for the equation $\dfrac{dy}{dx} = 1 + (x-y)(ax-by)$,

$$\text{or } \frac{dy}{dx} = 1 + ax^2 - (a+b)\,xy + by^2,$$

$q = (a+b)x, \quad r = -b$, and evidently $u = x$;

$$\therefore \int dx\,(q + 2ur) = \tfrac{1}{2}(a-b)\,x^2;$$

$$\therefore \frac{1}{y-x} = -be^{\frac{1}{2}(a-b)x^2} \times \int dx\,e^{-\frac{1}{2}(a-b)x^2}.$$

Ex. 5. $(1 - xy)\dfrac{dy}{dx} + y^2 + ax = 0.$

Assume $y = \dfrac{z - ax^2}{1 + xz}$, so that $z = \dfrac{y + ax^2}{1 - xy}$;

$$\text{then } \frac{1}{z^3 - a}\cdot\frac{dz}{dx} + \frac{x}{1 + ax^3} = 0,$$

where the variables are separated.

Ex. 6. $(y - x)\dfrac{dy}{dx} = \dfrac{n(1+y^2)^{\frac{3}{2}}}{\sqrt{1+x^2}}$; assume $y = \dfrac{x - z}{1 + xz}$:

$$\therefore \frac{z}{(1+z^2)\,(z + n\sqrt{1+z^2})}\,\frac{dz}{dx} - \frac{1}{1+x^2} = 0.$$

Ex. 7. $\dfrac{dy}{dx} = \dfrac{y\,(n + cx)}{y + a + bx + cx^2}$. Let $z = \dfrac{y\,(n+cx)}{y + a + bx + cx^2}$;

$$\therefore \frac{1}{(n-z)^2 - b\,(n-z) + ac}\cdot\frac{1}{z}\,\frac{dz}{dx} = \frac{1}{(a + bx + cx^2)\,(n + cx)}.$$

Ex. 8. $\dfrac{dy}{dx} + p + qy + ry^2 = 0$, where $q = \dfrac{d}{dx}\log\sqrt{\dfrac{r}{p}}$.

$$\therefore \left(\frac{dy}{dx} + p + ry^2\right)\sqrt{\frac{r}{p}} + y\,\frac{d}{dx}\sqrt{\frac{r}{p}} = 0,$$

$$\text{or } \frac{d}{dx}\left(y\sqrt{\frac{r}{p}}\right) + \left(1 + \frac{y^2 r}{p}\right)\sqrt{rp} = 0;$$

$$\therefore \tan^{-1}\left(y\sqrt{\frac{r}{p}}\right) + \int dx\,\sqrt{rp} = C,$$

$$\text{or } y = \sqrt{\frac{p}{r}}\tan\left(C - \int dx\,\sqrt{rp}\right).$$

Thus if we take $p = x^m$, $r = x^n$, then $q = \dfrac{n-m}{2x}$, and the solution of

$$\frac{dy}{dx} + x^m + \frac{1}{2x}(n-m)\,y + x^n y^2 = 0, \text{ is}$$

$$y = x^{\frac{m-n}{2}} \tan\left\{C - \frac{2x^{\frac{1}{2}(m+n+2)}}{m+n+2}\right\}.$$

Euler's Equation.

19. To integrate the equation

$$\frac{dy}{dx}\sqrt{a+bx+cx^2+ex^3+fx^4} + \sqrt{a+by+cy^2+ey^3+fy^4} = 0;$$

or, considering x and y as functions of a new variable t,

$$\frac{dy}{dt}\sqrt{X} + \frac{dx}{dt}\sqrt{Y} = 0.$$

Let the function of t which expresses x be determined by the equation $\dfrac{dx}{dt} = \sqrt{X}$, and therefore that which expresses y by the equation $\dfrac{dy}{dt} = -\sqrt{Y}$; also let $x + y = p$, $x - y = q$, p and q being functions of t.

Then since $\left(\dfrac{dx}{dt}\right)^2 = X$;

$$\therefore\ 2\frac{dx}{dt}\frac{d^2x}{dt^2} = \frac{dX}{dt}, \quad \text{or} \quad \frac{d^2x}{dt^2} = \tfrac{1}{2}\frac{dX}{dx};$$

$$\text{similarly } \frac{d^2y}{dt^2} = \tfrac{1}{2}\frac{dY}{dy}.$$

$$\therefore\ \frac{d^2p}{dt^2} = \tfrac{1}{2}\left(\frac{dX}{dx} + \frac{dY}{dy}\right)$$

$$= \tfrac{1}{2}\{2b + 2c(x+y) + 3e(x^2+y^2) + 4f(x^3+y^3)\}$$

$$= b + c(x+y) + \frac{3e}{4}\{(x+y)^2 + (x-y)^2\}$$

$$+ 2f(x+y)\frac{(x+y)^2+3(x-y)^2}{4}$$

$$= b + cp + \frac{3e}{4}(p^2+q^2) + \tfrac{1}{2}fp(p^2+3q^2),$$

and $\dfrac{dp}{dt}\cdot\dfrac{dq}{dt} = X - Y = b(x-y) + c(x^2-y^2) + e(x^3-y^3) + f(x^4-y^4)$

$$= bq + cpq + \frac{e}{4}q(3p^2+q^2) + \tfrac{1}{2}fpq(p^2+q^2);$$

$$\therefore \frac{d^2p}{dt^2} - \frac{1}{q}\frac{dp}{dt}\cdot\frac{dq}{dt} = \frac{e}{2}q^2 + fpq^2,$$

$$\text{or } \frac{d}{dt}\left(\frac{1}{q}\frac{dp}{dt}\right)^2 = e\frac{dp}{dt} + 2fp\frac{dp}{dt};$$

$$\therefore \left(\frac{1}{q}\frac{dp}{dt}\right)^2 = C + ep + fp^2;$$

$$\therefore \frac{dp}{dt} = q\sqrt{C+ep+fp^2},$$

$$\text{or } \sqrt{X} - \sqrt{Y} = (x-y)\sqrt{C+e(x+y)+f(x+y)^2},$$

the integral required. The discovery of this integral, which is due to Euler, was of great importance, as being the first step towards the foundation of the Theory of Elliptic Functions.

20. To integrate the equation

$$\sqrt{1-c^2\sin^2\psi} + \sqrt{1-c^2\sin^2\phi}\,\frac{d\psi}{d\phi} = 0,$$

or considering ϕ and ψ as functions of another variable t,

$$\sqrt{1-c^2\sin^2\psi}\,\frac{d\phi}{dt} + \sqrt{1-c^2\sin^2\phi}\,\frac{d\psi}{dt} = 0.$$

Let $\dfrac{d\phi}{dt} = \sqrt{1-c^2\sin^2\phi}$, and $\therefore\ \dfrac{d\psi}{dt} = -\sqrt{1-c^2\sin^2\psi}$;

$$\therefore \left(\frac{d\phi}{dt}\right)^2 = 1 - c^2\sin^2\phi, \quad \left(\frac{d\psi}{dt}\right)^2 = 1 - c^2\sin^2\psi;$$

$$\therefore \frac{d^2\phi}{dt^2} + \frac{d^2\psi}{dt^2} = -\tfrac{1}{2}c^2(\sin 2\phi + \sin 2\psi),$$

$$\frac{d^2\phi}{dt^2} - \frac{d^2\psi}{dt^2} = -\tfrac{1}{2}c^2(\sin 2\phi - \sin 2\psi).$$

Let $p = \phi + \psi, \qquad q = \phi - \psi,$

$$\therefore \frac{d^2p}{dt^2} = -c^2 \sin p \cos q,$$

$$\frac{d^2q}{dt^2} = -c^2 \cos p \sin q,$$

and $\dfrac{dp}{dt} \cdot \dfrac{dq}{dt} = \left(\dfrac{d\phi}{dt}\right)^2 - \left(\dfrac{d\psi}{dt}\right)^2 = -c^2(\sin^2\phi - \sin^2\psi)$

$$= \frac{c^2}{2}(\cos 2\phi - \cos 2\psi) = -c^2 \sin p \sin q;$$

$$\therefore \left(\frac{dp}{dt}\right)^{-1} \cdot \frac{d^2p}{dt^2} = \frac{\cos q}{\sin q}\frac{dq}{dt}; \qquad \therefore \log\left(\frac{dp}{dt}\right) = \log(C \sin q);$$

$$\therefore \frac{dp}{dt} = C \sin q; \quad \text{similarly } \frac{dq}{dt} = C \sin p;$$

$$\therefore \sqrt{1 - c^2 \sin^2\phi} \mp \sqrt{1 - c^2 \sin^2\psi} = C \sin(\phi \mp \psi)$$

is the integral of the proposed equation; which is only Euler's equation under a different form.

The equations $\sqrt{1 + y^4} + \sqrt{1 + x^4}\dfrac{dy}{dx} = 0,$

$$\sqrt{y - y^3} + \sqrt{x - x^3}\frac{dy}{dx} = 0,$$

are immediately reducible to the above form, viz.

$$\sqrt{1 - \tfrac{1}{2}\sin^2\psi} + \sqrt{1 - \tfrac{1}{2}\sin^2\phi}\frac{d\psi}{d\phi} = 0;$$

the former by making $x = \tan\frac{1}{2}\phi$, $y = \tan\frac{1}{2}\psi$; the latter by making $\sqrt{x} = \cos\phi$, $\sqrt{y} = \cos\psi$.

On the Factors which render integrable a Differential Equation of the First Order.

21. The most natural way of obtaining the complete integral of a differential equation of the first order, is to prepare it so that its first member may become an exact differential coefficient; for then we shall have only to integrate and add a constant. This preparation is always possible by means of a factor, when the equation is reduced to the form $\frac{dy}{dx} + K = 0$. For let an equation $f(x, y, C) = 0$ be resolved with respect to C, so that

$$C = \phi\,(x, y);$$

$$\therefore \text{ by differentiation, } 0 = P + Q\frac{dy}{dx}, \text{ or } \frac{dy}{dx} + \frac{P}{Q} = 0.$$

Now the equation $M + N\frac{dy}{dx} = 0$ may be put under the form $\frac{dy}{dx} + K = 0$, which agrees with the preceding, and may consequently be supposed to have arisen from the elimination of a constant between the primitive $f(x, y, C) = 0$, and its immediately derived equation. On this supposition, therefore, $\frac{dy}{dx} + K = 0$ is identical with $\frac{dy}{dx} + \frac{P}{Q} = 0$;

$$\therefore \frac{dy}{dx} + K = \frac{1}{Q}\left(P + Q\frac{dy}{dx}\right),$$

$$\text{or } \frac{d}{dx}\phi\,(x, y) = Q\left(\frac{dy}{dx} + K\right), \text{ identically.}$$

The second member therefore is an exact differential coefficient, which proves that there always exists a factor proper to render the expression $\frac{dy}{dx} + K$ integrable.

22. But although the existence of the factor in every case is thus established, the investigation of it is usually attended with greater difficulties than the solution of the original equation.

For let $P + Q\frac{dy}{dx} = 0$ be an exact differential equation; and let z, a function of x and y, be a common factor of P and Q so that $P = Mz$, $Q = Nz$, by the removal of which, the equation is reduced to the inexact state

$$M + N\frac{dy}{dx} = 0;$$

then because $P + Q\frac{dy}{dx} = 0$ is exact,

$$\frac{dP}{dy} = \frac{dQ}{dx}, \quad \text{or} \quad \frac{d(Mz)}{dy} = \frac{d(Nz)}{dx},$$

$$\text{or} \quad z\frac{dM}{dy} + M\frac{dz}{dy} = z\frac{dN}{dx} + N\frac{dz}{dx},$$

$$\text{or} \quad N\frac{dz}{dx} - M\frac{dz}{dy} = z\left(\frac{dM}{dy} - \frac{dN}{dx}\right);$$

an equation between x, y, z, and the partial differential coefficients of z, for determining the factor z. The consideration of this equation in its general state must be reserved till we come to treat of partial differential equations of the first order; but the following particular cases may be noticed.

23. First, suppose that the factor is a function of only one of the variables x, then $\frac{dz}{dy} = 0$, and the equation becomes

$$\frac{1}{z}\frac{dz}{dx} = \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right),$$

which, being integrated, gives z; for the hypothesis requires that the second member should be independent of y.

Similarly, if the factor be a function of y only, it will result from the integration of

$$\frac{1}{z}\frac{dz}{dy} = \frac{1}{M}\left(\frac{dN}{dx} - \frac{dM}{dy}\right),$$

of which the second member is independent of x.

Hence, if in any equation $M + N\frac{dy}{dx} = 0$ we find

$$\frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = X$$

a function of x only, or

$$\frac{1}{M}\left(\frac{dN}{dx} - \frac{dM}{dy}\right) = Y$$

a function of y only; the factors which make it integrable are respectively $e^{\int dx X}$, $e^{\int dy Y}$.

Ex. 1. $\frac{dy}{dx} + (Py - Q) = 0$, the linear equation of the first order.

This compared with $M + N\frac{dy}{dx} = 0$, gives

$$M = Py - Q, \qquad N = 1;$$

$$\therefore \frac{dM}{dy} - \frac{dN}{dx} = P, \quad \text{and } \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = P,$$

a function of x only; therefore the factor is $e^{\int dx P}$.

Ex. 2.
$$y^2 + (1 - xy)\frac{dy}{dx} = 0,$$

$$M = y^2, \quad N = 1 - xy,$$

$$\frac{dN}{dx} - \frac{dM}{dy} = -3y;$$

$$\therefore \int dy \frac{1}{M}\left(\frac{dN}{dx} - \frac{dM}{dy}\right) = -\int dy \frac{3}{y} = \log\frac{1}{y^3};$$

therefore the factor is $\frac{1}{y^3}$.

Ex. 3. $(y - x)\frac{dy}{dx} - \frac{1}{2cx} = 0$.

The factor by which this is made integrable is a function of both the variables $\frac{1}{x}e^{cy^2}$, as may be shewn by introducing it, and applying the criterion of integrability to the equation.

Ex. 4. $\frac{dy}{dx}+\frac{1}{q}\frac{dp}{dx}-\frac{1}{p}\frac{dq}{dx}y^2=0,$

where p and q are any functions of x, is made integrable by the factor $\frac{1}{(p+qy)^2}$, and its complete integral is

$$\frac{1}{q(p+qy)}+\int dx\frac{1}{pq^2}\frac{dq}{dx}=C.$$

24. In the case of homogeneous equations, a factor proper to render them integrable, is readily discovered by means of the property that if u be a homogeneous function of n dimensions of the independent quantities t and z, then

$$nu=t\frac{du}{dt}+z\frac{du}{dz}.$$

For suppose V, a homogeneous function of x and y of m dimensions, to be a factor which makes $M+N\frac{dy}{dx}$ an exact differential coefficient, M and N being homogeneous functions of x and y of r dimensions; then if U denote the primitive function, it will be homogeneous and of $m+r+1$ dimensions, and we shall have

$$VM+VN\frac{dy}{dx}=\frac{dU}{dx};$$

hence since VM and VN are the partial differential coefficients of U with respect to x and y respectively,

$$xVM+yVN=(m+r+1)U;$$

$$\therefore\ \frac{M+N\frac{dy}{dx}}{Mx+Ny}=\frac{1}{m+r+1}\cdot\frac{1}{U}\frac{dU}{dx},$$

and as the second member is an exact differential coefficient, it follows that the first is so likewise, and consequently, that $M+N\frac{dy}{dx}$ is made exact by means of the multiplier

$$\frac{1}{Mx+Ny}.$$

25. The property of homogeneous functions assumed above is easily proved. Let u be a homogeneous function of the independent quantities t and z of n dimensions; then if we change t into $t(1+h)$ and z into $z(1+h)$, u will become

$$u(1+h)^n = u + nuh + \&c.$$

But by Taylor's theorem, u will also become

$$u + \frac{du}{dt} \cdot ht + \frac{du}{dz} \cdot hz + \&c.,$$

therefore, equating the coefficients of h,

$$nu = t\frac{du}{dt} + z\frac{du}{dz}.$$

And, generally, if u be a homogeneous function of n dimensions of any number of independent quantities t, z, w, &c., and we change them into $t(1+h)$, $z(1+h)$, $w(1+h)$, &c., the new value of u will be equally expressed by $u(1+h)^n$ or by $e^{\left(t\frac{d}{dt}+z\frac{d}{dz}+\ldots\right)h}u$; and equating the coefficients of h^r in these two identical expressions, we get, separating as above the symbols of operation from those of quantity,

$$n(n-1)\ldots(n-r+1)\,u = \left(t\frac{d}{dt} + z\frac{d}{dz} + w\frac{d}{dw} + \ldots\right)^r u.$$

Ex. $xy + y^2 + (xy - x^2)\dfrac{dy}{dx} = 0.$

The factor is

$$\frac{1}{(xy+y^2)\,x + (xy-x^2)\,y} = \frac{1}{2y^2x};$$

$$\therefore\ \frac{xy+y^2}{2y^2x} + \frac{xy-x^2}{2y^2x}\frac{dy}{dx} = 0,$$

is an exact differential coefficient, and gives the primitive by Art. 7.

$$\frac{x}{2y} + \tfrac{1}{2}\log(xy) + C = 0.$$

26. Whenever the variables can be separated in an equation, a factor which makes it integrable can also be found.

For suppose that $M + N\dfrac{dy}{dx} = 0$, by the introduction of two other variables u and z, is transformed into $R + S\dfrac{du}{dz} = 0$, so that

$$M + N\frac{dy}{dx} = R + S\frac{du}{dz};$$

and suppose V to be a function of u and z, such that if we divide $R + S\dfrac{du}{dz}$ by it, the variables are separated, i. e. $\dfrac{R}{V}$ contains z only, and $\dfrac{S}{V}$ contains u only;

$$\therefore\ \frac{1}{V}\left(M + N\frac{dy}{dx}\right) = \frac{R}{V} + \frac{S}{V}\frac{du}{dz}$$

is an exact differential coefficient; and consequently $\dfrac{1}{V}$, which, upon restoring the values of u and z, becomes a function of x and y, is a factor which makes $M + N\dfrac{dy}{dx} = 0$ integrable.

Ex. 1. $a + bx^2y^2 + x^2\dfrac{dy}{dx} = 0.$

Assuming $y = \dfrac{u}{x}$, we find

$$a + bx^2y^2 + x^2\frac{dy}{dx} = a + bu^2 + x\frac{du}{dx} - u,$$

and dividing by $x(a - u + bu^2)$, we get

$$\frac{a + bx^2y^2 + x^2\dfrac{dy}{dx}}{x(a - u + bu^2)} = \frac{1}{x} + \frac{\dfrac{du}{dx}}{a - u + bu^2};$$

$$\therefore\ \frac{1}{x(a - u + bu^2)} = \frac{1}{ax - x^2y + bx^3y^2}$$

is a factor which makes the proposed equation integrable.

Ex. 2. $\frac{dy}{dx} + y^2 - \frac{a}{x^4} = 0.$

The integrating factor here is $\frac{x^2}{x^2(1-xy)^2 - a}$; as results from the substitution by which (Art. 16) the variables are separated.

Ex. 3. $y^2 + ax + (1-xy)\frac{dy}{dx} = 0.$

By assuming $y = \frac{z - ax^2}{1 + xz}$, it may be shewn that a factor which makes the proposed integrable is

$$\frac{1}{y^3 + 3ayx - a + a^2x^3}.$$

Ex. 4. $M + N\frac{dy}{dx} = 0$, a homogeneous equation.

In this case we know, that making $y = xz$, we have

$$M = x^r f(z), \quad N = x^r \phi(z),$$

$$\text{and } M + N\frac{dy}{dx} = x^r f(z) + x^r \phi(z)\left(z + x\frac{dz}{dx}\right);$$

consequently, dividing by

$$x^{r+1}\{f(z) + z\phi(z)\} = Mx + Ny,$$

we get

$$\frac{M + N\frac{dy}{dx}}{Mx + Ny} = \frac{1}{x} + \frac{\phi(z)\frac{dz}{dx}}{f(z) + z\phi(z)};$$

$$\therefore \frac{1}{Mx + Ny}$$

is a factor which makes the proposed equation integrable.

Obs. That $\frac{M + N\frac{dy}{dx}}{Mx + Ny}$ is an exact differential coefficient, provided M and N be homogeneous functions of x and y of the same dimensions, admits of an easy proof as follows.

We must shew that

$$\frac{d}{dy}\left(\frac{M}{Mx+Ny}\right)=\frac{d}{dx}\left(\frac{N}{Mx+Ny}\right).$$

Now putting $\frac{y}{x}=z$, we have

$$\frac{M}{Mx+Ny}=\frac{1}{x}\,\frac{1}{1+\frac{Ny}{Mx}}=\frac{1}{x}\cdot\frac{1}{1+F(z)},$$

$$\frac{N}{Mx+Ny}=\frac{1}{y}\left(1-\frac{Mx}{Mx+Ny}\right)=\frac{1}{y}-\frac{1}{y}\,\frac{1}{1+F(z)};$$

$$\therefore\ \frac{d}{dy}\left(\frac{M}{Mx+Ny}\right)=-\frac{1}{x}\cdot\frac{\frac{dF(z)}{dz}\cdot\frac{1}{x}}{\{1+F(z)\}^2},$$

$$\frac{d}{dx}\left(\frac{N}{Mx+Ny}\right)=-\frac{1}{y}\cdot\frac{\frac{dF(z)}{dz}\cdot\frac{y}{x^2}}{\{1+F(z)\}^2},$$

which expressions are evidently equal to one another.

27. There always exists an infinite number of factors which render an equation of the first order and degree integrable.

For let z be the factor by means of which the equation

$$M+N\frac{dy}{dx}=0$$

is made integrable, and $u=0$ its complete primitive, so that

$$z\left(M+N\frac{dy}{dx}\right)=\frac{du}{dx};$$

multiply both sides by $F(u)$, where $F(u)$ denotes any function of u, and we have

$$zF(u)\left(M+N\frac{dy}{dx}\right)=F(u)\,\frac{du}{dx};$$

and as the second member is an exact differential coefficient, it follows that the first is so likewise; therefore $zF(u)$ is a factor which makes the proposed equation integrable, whatever form be assigned to $F(u)$.

Geometrical Problems producing Equations of the first order and degree.

28. The following geometrical problems are added to illustrate this part of the subject.

I. To determine the trajectory of a given family of curves.

Let AA', BB' (fig. 1) be two of a family of curves resulting from the equation $f(X, Y, c) = 0$, by giving particular values to the constant c; and let AB be a curve which cuts AA', BB', and all the curves resulting from the equation by giving all possible values to c, at the same angle; then AB is called the trajectory of this family of curves. Let x, y, be the co-ordinates of the point A in AB, between which we are required to find a relation; AT', AT, tangents to the curve and trajectory at A, $\tan TAT' = a$; and let a value $\psi(X, Y)$ of $\frac{dY}{dX}$ be obtained from the equation $f(X, Y, c) = 0$, not involving c; then at the point A, $\tan AT'N = \psi(x, y)$, and $\tan ATN = \frac{dy}{dx}$,

$$\therefore \; a = \frac{\frac{dy}{dx} - \psi(x, y)}{1 + \frac{dy}{dx} \cdot \psi(x, y)},$$

the differential equation to the curve AB; and as it does not involve c, AB will cut every curve in the series at an angle whose tangent $= a$. The equation when integrated will involve an arbitrary constant, and consequently will represent a system of curves, every one of which cuts the former system at the same angle; the constant may be determined, if a point through which the trajectory is to pass, be given. If the angle TAT' be a right angle, or a be infinite, the differential equation to the trajectory, which is then called orthogonal, is

$$1 + \frac{dy}{dx} \cdot \psi(x, y) = 0.$$

Ex. 1. To find the orthogonal trajectory to all curves resulting from the equation $y^2(c - x) = x^3$, by giving all possible values to c.

$$\text{Here } c-x=\frac{x^3}{y^2}, \qquad \therefore\ -1=\frac{3x^2}{y^2}-\frac{2x^3}{y^3}\frac{dy}{dx};$$

$$\therefore\ \frac{dy}{dx}=\psi(x, y)=\frac{y^3+3x^2y}{2x^3},$$

therefore, substituting for $\psi(x, y)$ its value, the differential equation to the trajectory is

$$2x^3+(y^3+3x^2y)\frac{dy}{dx}=0;$$

a homogeneous equation whose integral is (Art. 11)

$$x^2+y^2=C\sqrt{2x^2+y^2}.$$

Similarly, let $f(\rho, \theta, c)=0$ be the equation to the curve AA' referred to polar co-ordinates, and let it give for $\rho\frac{d\theta}{d\rho}$ the value $\psi(\rho, \theta)$ independent of c; then considering ρ and θ as co-ordinates of the point A in the curve AB,

$$\tan SAT'=\psi(\rho, \theta), \quad \tan SAT=\rho\frac{d\theta}{d\rho},$$

$$\therefore\ a=\frac{\psi(\rho, \theta)-\rho\dfrac{d\theta}{d\rho}}{1+\psi(\rho, \theta)\,\rho\dfrac{d\theta}{d\rho}},$$

which is the differential equation to the trajectory; or if it be orthogonal,

$$1+\psi(\rho, \theta)\,\rho\frac{d\theta}{d\rho}=0.$$

Ex. 2. Let the curves be a system of circles touching a straight line in the same point, then taking that point as the origin and measuring θ from the line, their equation is

$$\rho=c\sin\theta,$$

$$\therefore\ \frac{1}{\rho}=\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\rho}, \quad \text{or } \rho\frac{d\theta}{d\rho}=\frac{\sin\theta}{\cos\theta}=\psi(\rho, \theta);$$

$$\therefore\ 1+\frac{\sin\theta}{\cos\theta}\rho\frac{d\theta}{d\rho}=0, \quad \text{or } \cos\theta+\sin\theta\,\rho\frac{d\theta}{d\rho}=0,$$

$$\text{or } \frac{d}{d\rho}\left(\frac{\rho}{\cos\theta}\right) = 0; \quad \therefore \rho = C\cos\theta,$$

the equation to the orthogonal trajectory, which represents a system of circles passing through the given point and having the given line for their diameter.

We may generalize this problem, by finding the orthogonal trajectory of all circles described through two given points.

II. To determine a curve such, that the locus of the extremity of its polar subtangent shall be a straight line.

The polar subtangent is a line drawn from the origin perpendicular to the radius vector to meet the tangent.

Let ρ, θ, be the polar co-ordinates of any point P in the curve sought (fig. 2); then those of the extremity T of its polar subtangent will be $\rho^2\frac{d\theta}{d\rho}$ and $\theta - \frac{\pi}{2}$, which must satisfy the equation to a straight line, viz.

$$\rho' = c\sec(\theta' - \alpha);$$

$$\therefore \rho^2\frac{d\theta}{d\rho} = c\sec\left(\theta - \alpha - \frac{\pi}{2}\right) = \frac{c}{\sin(\theta - \alpha)}.$$

$$\therefore \frac{c}{\rho^2} = \sin(\theta - \alpha)\frac{d\theta}{d\rho}; \quad \text{and } \frac{c}{\rho} = \cos(\theta - \alpha) + C;$$

$$\therefore \rho = \frac{c}{C + \cos(\theta - \alpha)},$$

the equation to curves of the second order, having the pole for one of their foci.

III. To find a curve in which SG varies as SP, PG being a normal at P, and SG a fixed line through S, (fig. 2).

Taking SG for the axis of x, the equation to the normal at P is

$$(Y - y)\frac{dy}{dx} + X - x = 0;$$

therefore making $\quad Y = 0, \quad X = SG = x + y\frac{dy}{dx},$

$$\text{and } SP=\sqrt{x^2+y^2}, \quad \therefore\ x+y\frac{dy}{dx}=e\sqrt{x^2+y^2},$$

$$\text{or } \sqrt{x^2+y^2}=ex+C,$$

the equation to a curve of the second order.

IV. To find a curve which is always cut by its radius vector at an angle proportional to the corresponding angle of revolution; that is, $\angle SPT \propto \angle ASP$, (fig. 2).

Let ρ, θ, be the co-ordinates of any point in the curve, then the angle at which the radius vector cuts the curve, has for its tangent $\rho\dfrac{d\theta}{d\rho}$;

$$\therefore\ \rho\frac{d\theta}{d\rho}=\tan n\theta, \quad \text{or } \frac{1}{\rho}\frac{d\rho}{d\theta}=\frac{\cos n\theta}{\sin n\theta}, \quad \therefore\ \left(\frac{\rho}{C}\right)^n=\sin n\theta.$$

V. To find the locus of the centre of an ellipse rolling along a straight line.

Let ON be the line along which the ellipse rolls, and which touches the ellipse at P; $ON=x$, $NC=y$, the co-ordinates of its centre; then CP is a normal to the locus of C (fig. 7), and therefore

$$=y\sqrt{1+\left(\frac{dy}{dx}\right)^2},$$

$$\text{but} \quad a^2+b^2-CP^2=\frac{a^2b^2}{CN^2},$$

$$\therefore\ a^2+b^2-y^2-y^2\left(\frac{dy}{dx}\right)^2=\frac{a^2b^2}{y^2}, \quad \text{or } \frac{dx}{dy}=\frac{y^2}{\sqrt{(a^2-y^2)(y^2-b^2)}}$$

the differential equation to the required curve, in which the variables are separated. Generally, if $p=f(\rho)$ be the relation between the radius vector and perpendicular on the tangent in any curve, then the locus of the pole, when the curve rolls along a straight line, will have for its equation

$$y=f\left\{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right\}.$$

Thus if the curve be a parabola, to find the locus of its focus we have $p = \sqrt{a\rho}$;

$$\therefore\ y = \sqrt{ay\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}, \quad \text{or } y = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

the equation to the common catenary.

VI. To find a curve such that the intersection of the tangent at any point and a line drawn from the pole inclined at a constant angle to the radius vector of that point, shall trace out a given curve.

Let $\angle ASP = \theta$, $SP = \rho$ (fig. 2) be the polar co-ordinates of any point P in the required curve, YQ a tangent at P intersected by a perpendicular upon it from S in Y, and by a line through S inclined at a constant angle $PSQ = \alpha$ to SP in Q. Also let $\angle ASQ = \theta'$, $SQ = \rho'$ be the co-ordinates of Q, which is supposed to lie in a given curve whose equation is $\frac{1}{\rho'} = f(\theta')$; then $\theta' = \theta + \alpha$; and, calling $\angle PSY = \phi$,

$$\frac{1}{\rho'} = \frac{\cos(\alpha + \phi)}{\rho \cos \phi} = \frac{\cos \alpha}{\rho} - \frac{\sin \alpha}{\rho^2}\frac{d\rho}{d\theta}, \text{ since } \cot \phi = \frac{1}{\rho}\frac{d\rho}{d\theta}.$$

Hence the equation to the locus of P is

$$\frac{d}{d\theta}\left(\frac{1}{\rho}\right) + \frac{\cot \alpha}{\rho} = \frac{1}{\sin \alpha} f(\theta + \alpha),$$

$$\text{or } \frac{1}{\rho} e^{\theta \cot \alpha} + C = \frac{1}{\sin \alpha} \int d\theta\, e^{\theta \cot \alpha} f(\theta + \alpha).$$

Suppose $\frac{a}{\rho'} = \frac{\cos \beta + c \cos \theta'}{a}$, a curve of the second order; then $\frac{a}{\rho} + Ce^{-\theta \cot \alpha} = \frac{\cos \beta}{\cos \alpha} + c \cos \theta$, another curve of the second order when $C = 0$. The result seems to fail when $\alpha = \frac{1}{2}\pi$; but in that case, first changing the constant into $C' + \frac{\cos \beta}{\cos \alpha}$, we find

$$\frac{a}{\rho} + C' - \theta \cos \beta = c \cos \theta.$$

SECTION II.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER BUT NOT OF THE FIRST DEGREE.

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29. WHEN a differential equation of the first order is of a higher degree than the first, we know that it is not obtained by the direct differentiation of its primitive, but results from eliminating a constant, (which enters into the primitive in a dimension above the first,) between the primitive and its derived equation; the degree of the differential equation and the dimension of the constant eliminated above the lowest dimension in which it appears, being always the same. The general form of such equations free from radicals, is

$$\left(\frac{dy}{dx}\right)^n + p_1\left(\frac{dy}{dx}\right)^{n-1} + p_2\left(\frac{dy}{dx}\right)^{n-2} + \dots + p_{n-1}\frac{dy}{dx} + p_n = 0,$$

the coefficients being functions of x and y.

If this can be resolved with respect to $\frac{dy}{dx}$ into its n simple factors, it will assume the form

$$\left(\frac{dy}{dx} + q_1\right)\left(\frac{dy}{dx} + q_2\right) \dots \left(\frac{dy}{dx} + q_n\right) = 0;$$

then each of these factors put equal to zero, will be an equation of the first order and degree, whose integral may be found by the methods of the preceding Section; and any one of these integrals, as well as the continued product of any number of them, will evidently satisfy the proposed equation. If, therefore, we integrate the n equations,

$$\frac{dy}{dx} + q_1 = 0, \quad \frac{dy}{dx} + q_2 = 0, \dots \quad \frac{dy}{dx} + q_n = 0,$$

and complete them all with the same constant C, as the proposed equation is of the first order, we shall obtain the required primitive involving C in the n^{th} power, by equating their continued product to zero.

Ex. 1. $\left(\frac{dy}{dx}\right)^2 + \frac{2x}{y}\frac{dy}{dx} - 1 = 0$;

$$\therefore \frac{dy}{dx} + \frac{x}{y} = \pm\sqrt{1+\frac{x^2}{y^2}}, \quad \text{or} \quad \frac{y\frac{dy}{dx}+x}{\sqrt{y^2+x^2}} = \pm 1;$$

$$\therefore +\sqrt{y^2+x^2} = x + C, \text{ and } -\sqrt{y^2+x^2} = x + C;$$

$$\therefore (\sqrt{y^2+x^2} - C - x)(\sqrt{y^2+x^2} + C + x) = 0,$$

$$\therefore y^2 + x^2 - (C+x)^2 = 0, \text{ or } y^2 = 2Cx + C^2.$$

Ex. 2. $x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2 - x^2y^2 - x^4 = 0.$

$$\therefore x\frac{dy}{dx} - y = \pm x\sqrt{x^2+y^2}, \quad \text{or} \quad \frac{\frac{d}{dx}\left(\frac{y}{x}\right)}{\pm\sqrt{1+\frac{y^2}{x^2}}} = 1.$$

$$\therefore \log\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) = x + C,$$

$$\text{and } -\log\left(\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right) = \log\left(\sqrt{1+\frac{y^2}{x^2}} - \frac{y}{x}\right) = x + C;$$

$$\therefore \sqrt{x^2+y^2} + y = cxe^x, \text{ changing the constant,}$$

$$\text{and } \sqrt{x^2+y^2} - y = cxe^x;$$

$$\therefore (\sqrt{x^2+y^2} - cxe^x + y)(\sqrt{x^2+y^2} - cxe^x - y) = 0,$$

$$\text{or } y = \frac{x}{2}\left(\frac{1}{ce^x} - ce^x\right).$$

Ex. 3. $y^2 + \left(y\frac{dy}{dx}\right)^2 = 2nx + 2ny\frac{dy}{dx}.$

$$y^2 + (x-c)^2 = n^2 + 2nx,$$

the equation to a circle.

This is the solution of the Problem to find a curve in which the square of the normal is always proportional to the sum of the abscissa and subnormal.

30. When the resolution of the proposed equation into its simple factors is impossible, there are still various forms for which the complete primitive can be determined, or its determination made to depend on elimination; this is done by means of substitution, or differentiation, or other analytical artifices, of which we shall now give some instances.

Obs. The arbitrary constant in what follows is often reserved under sign of integration.

31. When the equation contains only one of the variables, x suppose, and can be solved with respect to that variable, so that $x = f\left(\frac{dy}{dx}\right)$; let $\frac{dy}{dx}$ be denoted by p, then $x = f(p)$; and integrating the equation $\frac{dy}{dx} = p$ by parts, we get

$$y = xp - \int dx \frac{dp}{dx} = pf(p) - \int dp f(p);$$

between which and the equation $x = f(p)$, eliminating p, we shall obtain the required integral.

Similarly, if we have $y = f(p)$, since

$$\frac{dx}{dp} = \frac{dx}{dy} \cdot \frac{dy}{dp} = \frac{1}{p} \frac{df(p)}{dp},$$

we shall have to eliminate p between $y = f(p)$,

$$\text{and } x = \int dp \frac{1}{p} \frac{df(p)}{dp}.$$

Ex. 1. $x + x\left(\frac{dy}{dx}\right)^2 = 1$, or $x = \frac{1}{1+p^2}$;

$$\therefore\ y = \frac{p}{1+p^2} - \tan^{-1} p + C;$$

$$\therefore\ y = \sqrt{x(1-x)} - \tan^{-1} \sqrt{\frac{1-x}{x}} + C.$$

Ex. 2. $y = a\sqrt{1+p^2}, \quad x + C = a \log (\sqrt{y^2 - a^2} + y).$

This is the solution of the problem in which it is required to find a curve such that the perpendicular on the tangent from the foot of the ordinate shall be constant.

Ex. 3. $y\sqrt{1+p^2} = ap.$ Here $\dfrac{dy}{dp} = \dfrac{d}{(1+p^2)^{\frac{3}{2}}}$;

$$\therefore\ x + C = \int dp \frac{a}{p(1+p^2)^{\frac{3}{2}}} = \frac{a}{\sqrt{1+p^2}} + a \log \left(\frac{p}{\sqrt{1+p^2}+1}\right),$$

$$\text{or } x + C = \sqrt{a^2 - y^2} + a \log \left(\frac{y}{\sqrt{a^2 - y^2} + a}\right).$$

This is the solution of the problem to find a curve such that the tangent terminated at the axis of x shall be of a constant length.

32. An equation not coming immediately under this case, may sometimes be reduced to it by putting $p = xz$, or $p = yz$.

Ex. $\left(\dfrac{dy}{dx}\right)^3 + ax\dfrac{dy}{dx} + x^3 = 0.$ Let $p = xz$,

then $x(z^3+1) + az = 0$, or $x = -\dfrac{az}{1+z^3}$;

$$\therefore\ \frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = -\frac{a^2 z^2 (2z^3 - 1)}{(z^3+1)^3},$$

and z must be eliminated between the integral of this, and the equation $x = -\dfrac{az}{1+z^3}$.

33. When the equation contains both the variables x and y, provided it be homogeneous with respect to them, we may assume $\dfrac{y}{x} = z$; then the equation will take the form, (which is not solvable with respect to p by supposition,)

$$f(z, p) = 0.$$

Suppose this capable of being solved with respect to z, and let it give $z=\phi(p)$; now $y=xz$ gives $p=z+x\dfrac{dz}{dx}$,

$$\text{or } \frac{1}{x}=\frac{1}{p-z}\frac{dz}{dx},$$

substitute the above value of z, and integrate this equation; then p must be eliminated between the result which will be of the form $\log x=F(p)$, and $y=x\phi(p)$.

Ex. $$y-x\frac{dy}{dx}=nx\sqrt{1+\left(\frac{dy}{dx}\right)^2};$$

$$\therefore \frac{y}{x}=p+n\sqrt{1+p^2};$$

$$\therefore \frac{1}{x}=\frac{\dfrac{d}{dx}(p+n\sqrt{1+p^2})}{-n\sqrt{1+p^2}}=-\frac{1}{n}\left(\frac{\dfrac{dp}{dx}}{\sqrt{1+p^2}}+n\frac{\dfrac{d}{dx}\sqrt{1+p^2}}{\sqrt{1+p^2}}\right);$$

$$\therefore \log x=-\frac{1}{n}\{\log(p+\sqrt{1+p^2})+n\log\sqrt{1+p^2}\}+\log C;$$

$$\therefore x=\frac{C(p+\sqrt{1+p^2})^{-\frac{1}{n}}}{\sqrt{1+p^2}},\quad \frac{y}{x}=p+n\sqrt{1+p^2},$$

between which equations p must be eliminated.

This is the solution of the problem, to find a curve such that the perpendicular upon the tangent from the origin shall vary as the abscissa to the point of contact.

Let $n=1$. $$\frac{C}{x}=1+p^2+p\sqrt{1+p^2},\quad \frac{y}{x}=p+\sqrt{1+p^2};$$

$$\therefore \frac{y^2+(C-x)^2}{x^2}=(1+p^2)(p+\sqrt{1+p^2})^2=\frac{C^2}{x^2},$$

or $y^2=2Cx-x^2$, the equation to a circle.

34. Another integrable form is $y=x\dfrac{dy}{dx}+f\left(\dfrac{dy}{dx}\right)$, which

is called Clairaut's form, after the Mathematician who first considered it. Substituting p for $\frac{dy}{dx}$, and differentiating, we get successively,

$$y = xp + f(p),$$

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{d}{dp}f(p) \cdot \frac{dp}{dx};$$

$$\therefore \ \left\{x + \frac{d}{dp}f(p)\right\}\frac{dp}{dx} = 0,$$

which resolves itself into the two

$$x + \frac{d}{dp}f(p) = 0, \quad \frac{dp}{dx} = 0.$$

The first of these gives $p = \phi(x)$ suppose; this value substituted for p in the proposed equation, furnishes a relation between x and y which satisfies the proposed equation, but which involves no arbitrary constant, and cannot therefore be the complete primitive. The other equation must therefore lead to the complete primitive; but this gives $p = C$, and by substituting this value of p in the proposed we find

$$y = Cx + f(C).$$

Hence Clairaut's form has the property, that the complete primitive is obtained by substituting the arbitrary constant C for p, in that form. If we integrate $p = C$, we find $y = Cx + C'$; but the condition of the proposed equation being satisfied gives $C' = f(C)$, the same result as before.

We shall afterwards return to the consideration of the other solution, which is called the singular solution, and is not derivable from the complete integral. It is evident that Clairaut's form may be put into the rather more general shape

$$y = (x + c)p + f(p).$$

Ex. 1. $y = xp + \dfrac{a(1+p^2)}{p}$.

Differentiating, we get $p = p + x\dfrac{dp}{dx} + a\left(1 - \dfrac{1}{p^2}\right)\dfrac{dp}{dx}$,

$$\text{or}\ \left(x + a - \frac{a}{p^2}\right)\cdot\frac{dp}{dx} = 0;$$

$$\therefore\ \frac{dp}{dx} = 0 \text{ gives } p = C, \text{ and } y = Cx + \frac{a(1 + C'^2)}{C},$$

the complete integral; and

$$x + a - \frac{a}{p^2} = 0 \text{ gives } p = \pm\sqrt{\frac{a}{x+a}},$$

which, substituted in $py = a + p^2(x + a)$, gives the singular solution

$$\pm y\sqrt{\frac{a}{x+a}} = 2a, \text{ or } y^2 = 4a(x + a).$$

Ex. 2. $y = x(p - c) + \sqrt{a^2 - c^2 + p^2a^2}$,

$$y = x(C - c) + \sqrt{a^2 - c^2 + C^2a^2}, \text{ the complete integral;}$$

$$\frac{y^2}{a^2 - c^2} + \frac{(x - c)^2}{a^2} = 1, \text{ the singular solution.}$$

The two foregoing examples are the solutions of the problems to find a curve for which the locus of the intersection of the tangent and perpendicular upon it from the origin shall be a straight line, and a circle, respectively. For the co-ordinates of the said point of intersection are

$$X = -\frac{p(y - px)}{1 + p^2}, \quad Y = \frac{y - px}{1 + p^2};$$

and these substituted in the equations to a straight line and circle, viz. $X + a = 0$, $Y^2 + (X - c)^2 = a^2$, lead to the preceding equations.

Ex. 3. To find a curve such that, α, β, γ being the angles which the sides of a triangle on a given base and with its vertex a point in the curve, form with the tangent at that point, we may always have $\tan^2\beta = \tan\alpha\tan\gamma$.

Let the given base $SC=c$, $SN=x$, $NP=y$, the co-ordinates of a point P in the curve, PT a tangent, PG a normal at that point, AP parallel to SC; $\angle APT=\alpha$, $SPT=\beta$, $CPT=\gamma$ (fig. 8); then $\tan^2\beta=\tan\alpha\tan\gamma$ gives

$$1-\frac{\cos^2\beta}{\cos^2\alpha}=\tan\alpha\tan(\gamma-\alpha);$$

$$\therefore\ 1-\left(\frac{SG}{SP}\right)^2=\frac{GN}{PN}\cdot\frac{PN}{CN}=\frac{GN}{CN};$$

$$\text{or } 1-\frac{\left(x+y\dfrac{dy}{dx}\right)^2}{x^2+y^2}=-\frac{y}{x-c}\,\frac{dy}{dx};$$

$$\text{or}\qquad \frac{x+y\dfrac{dy}{dx}-c}{x-c}=\frac{\left(x+y\dfrac{dy}{dx}\right)^2}{x^2+y^2};$$

$$\text{or}\qquad \frac{\rho\dfrac{d\rho}{dx}-c}{x-c}=\left(\frac{d\rho}{dx}\right)^2;\ \text{or, putting } \frac{d\rho}{dx}=p,$$

$$\rho=(x-c)p+\frac{c}{p};$$

$$\therefore\ p=p+(x-c)\frac{dp}{dx}-\frac{c}{p^2}\frac{dp}{dx};\quad \therefore\ \frac{dp}{dx}=0,\ \text{or } p=C;$$

$$\therefore\ \sqrt{x^2+y^2}=(x-c)\,C+\frac{c}{C},\ \text{the complete integral,}$$

representing a conic Section, the two fixed points being its focus and center, and its eccentricity $=C$; and $(x-2c)^2+y^2=0$, is the singular solution, representing a point, viz. the other focus.

Ex. 4. $y=xp+\sqrt{b^2+a^2p^2}$,

$$y=Cx+\sqrt{b^2+C^2a^2},\ \text{the complete integral,}$$

$$a^2y^2+b^2x^2=a^2b^2,\ \text{the singular solution.}$$

This is the solution of the problem to find a curve, such that the product of the perpendiculars dropped from two given

points upon the tangent may be invariable; for taking the line joining the two given points (whose distance suppose $= 2c$) for the axis of x, and their middle point for origin, and x, y the co-ordinates of any point in the curve, the equation to the tangent at that point will be

$$Y - y = \frac{dy}{dx}(X - x), \quad \text{or } Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right),$$

and the lengths of the perpendiculars dropped upon this line from the points (c, o), $(-c, o)$ will be

$$\frac{-pc - (y - xp)}{\sqrt{1+p^2}}, \quad \frac{pc - (y - xp)}{\sqrt{1+p^2}};$$

$$\text{and the product of these} = \frac{(y - xp)^2 - c^2p^2}{1+p^2} = b^2, \text{ suppose;}$$

$$\therefore\ y = xp + \sqrt{b^2 + a^2p^2}, \text{ putting } a^2 = b^2 + c^2.$$

35. A still more general case is the equation

$$y = xf(p) + \phi(p),$$

which by differentiation is reduced to a linear equation of the first order in x; for we get

$$p = f(p) + x\frac{d}{dp}f(p)\frac{dp}{dx} + \frac{d}{dp}\phi(p)\frac{dp}{dx};$$

$$\therefore\ \{p - f(p)\}\frac{dx}{dp} = x\frac{d}{dp}f(p) + \frac{d}{dp}\phi(p);$$

$$\therefore\ \frac{dx}{dp} + x\frac{\frac{d}{dp}f(p)}{f(p) - p} = -\frac{\frac{d}{dp}\phi(p)}{f(p) - p},$$

which gives $x = F(p)$; then p must be eliminated between this and the proposed equation.

Ex. 1. $y = xp^2 + 2p$; $\therefore\ p = p^2 + 2xp\dfrac{dp}{dx} + 2\dfrac{dp}{dx}$,

$$\text{or } \frac{dx}{dp} + \frac{2x}{p-1} = -\frac{2}{p^2 - p},$$

which is made integrable by the factor $(p-1)^2$;

$$\therefore\ x\,(p-1)^2 = -\int dp\,\frac{2p-2}{p} = -2p + \log p^2 + C.$$

But $p = -\dfrac{1}{x} \pm \sqrt{\dfrac{y}{x} + \dfrac{1}{x^2}}$; therefore substituting this in the preceding, we obtain the complete primitive between x and y.

Ex. 2. $y = xmp + n\sqrt[3]{1+p^3}$;

$$xp^{\frac{m}{m-1}} = -\frac{n}{m-1}\int dp\,\frac{p^{\frac{2m-1}{m-1}}}{(1+p^3)^{\frac{2}{3}}}.$$

Ex. 3. $y + p\,(a-x) = n\int dx\sqrt{1+p^2}$;

$$y = \frac{(a-x)^{n+1}}{2c^n\,(n+1)} + \frac{c^n}{2\,(n-1)\,(a-x)^{n-1}} + C'.$$

This is the solution of the problem of finding the path of a point P which moves uniformly towards another point Q also moving uniformly in a straight line.

For taking A (fig. 3) for the origin, and AB, which is perpendicular to By the line in which Q moves, for the axis of x, we have, supposing P and Q to start together from A and B, $BQ = nAP$, or if $AN = x$, $NP = y$, $AB = a$,

$$y + (a-x)\,p = n\int dx\sqrt{1+p^2}.$$

36. In the following examples the method of substitution succeeds.

Ex. 1. $(1-p^2)\,xy = p\,(x^2 - y^2 - c^2)$,

which expresses that the normal bisects the angle between the focal distances; $2c$ being the distance of the foci, the origin at the middle point between them, and the line joining them the axis of x.

$$\text{Let } p = \frac{xz}{y}, \quad \therefore\ y^2 = x^2 z - c^2\,\frac{z}{1+z};$$

therefore, differentiating,

$$\left\{x^2 - \frac{c^2}{(1+z)^2}\right\}\frac{dz}{dx} = 0;$$

this resolves itself into

$$x = \pm \frac{c}{1+z}, \text{ which gives } y^2 + (x-c)^2 = 0,$$

the singular solution;

and $\frac{dz}{dx} = 0$, or $z = C$, which gives $y^2 = C\left(x^2 - \frac{c^2}{1+C}\right)$, the complete integral.

If we integrate $p = \frac{Cx}{y}$, we get $y^2 = Cx^2 + C'$, where C' must be determined by the condition of the proposed equation being satisfied; and by this condition, in general whenever the method of solution raises the order of the equation, must the number of constants be reduced.

By the same substitution may be solved the more general form

$$axyp^2 + p(x^2 - ay^2 - b) - xy = 0.$$

Ex. 2. $x\frac{dy}{dx} - y = X\sqrt{\left(\frac{dy}{dx}\right)^2 - \frac{2y}{x}\frac{dy}{dx} + 1}$,

$$\text{or } \frac{dy}{dx} - \frac{y}{x} = \frac{X}{x}\sqrt{\left(\frac{dy}{dx} - \frac{y}{x}\right)^2 + 1 - \frac{y^2}{x^2}}.$$

Let $y = xz$, then $\frac{dy}{dx} = z + x\frac{dz}{dx} = \frac{y}{x} + x\frac{dz}{dx}$;

$$\therefore\ x^2\left(\frac{dz}{dx}\right)^2 = \frac{X^2}{x^2}\left\{x^2\left(\frac{dz}{dx}\right)^2 + 1 - z^2\right\};$$

$$\therefore\ \frac{1}{\sqrt{1-z^2}}\frac{dz}{dx} = \frac{X}{x\sqrt{x^2 - X^2}}.$$

If $X = 1$, we have $\sin^{-1} z = \sec^{-1} x + C$,

$$\text{or } \sin^{-1}\frac{y}{x} = \sec^{-1} x + C.$$

Ex. 3 and 4.

$$\frac{y - xp}{\sqrt{1+p^2}} = f(\sqrt{x^2 + y^2}),$$

$$\frac{y - xp}{\sqrt{x^2+y^2}\sqrt{1+p^2}} = f\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

Introducing polar co-ordinates, we get for the first,

$$\frac{\rho^2}{\sqrt{\left(\frac{d\rho}{d\theta}\right)^2+\rho^2}}=f(\rho), \quad \text{or } \frac{d\theta}{d\rho}=\frac{f(\rho)}{\rho\sqrt{\rho^2-\{f(\rho)\}^2}}.$$

The second gives

$$\frac{\rho}{\sqrt{\left(\frac{d\rho}{d\theta}\right)^2+\rho^2}}=f(\cos\theta), \quad \text{or } \frac{1}{\rho}\frac{d\rho}{d\theta}=\frac{\sqrt{1-\{f(\cos\theta)\}^2}}{f(\cos\theta)}.$$

The former expresses that the perpendicular on the tangent from the origin is a given function of the radius vector; the latter that the sine of the angle at which the radius vector cuts the curve is a given function of the cosine of the angle at which it is inclined to the axis of x.

SECTION III.

ON THE SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

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37. FROM the complete integral of a differential equation we can deduce as many particular integrals as we please, by giving to the arbitrary constant particular values. But some differential equations are satisfied by a relation between x and y not containing an arbitrary constant, and not deducible from the complete integral; such a relation is called, as has been said, a singular solution of the differential equation. The existence of such solutions depends upon the fact, that when a solution of a differential equation has been obtained in any manner, it will still be a solution after a quantity of any kind has been introduced in any way, provided the same derived equation result. This is merely an extension of the principle on which the arbitrary constant is added.

38. Before entering upon the general theory, it may be useful to consider the following particular instance.

Let the equation

$$y = x\frac{dy}{dx} + a + a\left(\frac{dy}{dx}\right)^2 \dots\dots\dots(1)$$

be proposed, which, since it falls under Clairaut's form, has for its complete integral

$$y = Cx + a + aC^2; \dots\dots\dots(2)$$

C being the arbitrary constant. If we now regard C, not as a constant, but as a function of x, and differentiate, we get

$$\frac{dy}{dx} = C + (x + 2aC)\frac{dC}{dx};$$

and if we eliminate C between this and $y = Cx + a + aC^2$, we shall obtain a differential equation, but not the proposed one,

for that arises by eliminating C by means of the equation $\frac{dy}{dx} = C$. But if C be so determined as to make the coefficient of $\frac{dC}{dx}$ vanish, that is, if $C = -\frac{x}{2a}$, then the derived equation will be $\frac{dy}{dx} = C$, and the result of the elimination of C will be the proposed equation.

Substituting for C its value, we get

$$y = -\frac{x^2}{2a} + a + \frac{x^2}{4a} = -\frac{x^2}{4a} + a,$$

a relation which manifestly satisfies the proposed equation; for it gives $\frac{dy}{dx} = -\frac{x}{2a}$; and these values of y and $\frac{dy}{dx}$, being substituted in the proposed equation, make it identical. But this solution contains no arbitrary constant, and being the equation to a parabola it cannot, either by making $C = 0$, or any other constant quantity, arise from the complete integral which is the equation to a straight line; it is consequently a singular solution, and arises from the complete integral by changing C into a function of x so determined as to make the term involving $\frac{dC}{dx}$ disappear from the value of $\frac{dy}{dx}$.

Thus we see how the singular solution arises from the complete integral; next, let us consider its geometrical signification. The proposed differential equation expresses that the curves to which it belongs have the property, that the tangent at any point is intersected by a perpendicular upon it from a given point, in a given straight line.

Take the given point S (fig. 4) for the origin, and TS, AS respectively parallel and perpendicular to the given line AC, for the axes of x and y. Let TC be the tangent at a point whose co-ordinates are x and y, then its equation is

$$Y - y = \frac{dy}{dx}(X - x),$$

and the equation to the perpendicular upon it from S is

$$Y\frac{dy}{dx} = -X,$$

and for their point of intersection

$$Y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} - y = -x\frac{dy}{dx},$$

but this point is always in AC for which $Y = AS = a$;

$$\therefore\ y - x\frac{dy}{dx} = a + a\left(\frac{dy}{dx}\right)^2,$$

the same as the proposed equation.

The complete integral $y = Cx + a(1 + C^2)$, which represents a series of straight lines, evidently satisfies the problem for all values of C; for let Tb be any one of these lines, then $y = -\frac{x}{C}$ is the equation to a line through S perpendicular to it; and combining the equations to get the co-ordinates of their point of intersection, we have

$$y = -C^2y + a(1 + C^2), \quad \text{or } y = a;$$

the intersection consequently falls in AC. And as a straight line is its own tangent at every point, the equation

$$y = Cx + a(1 + C^2), \text{ for all values of } C,$$

represents a line such that the intersection of the tangent at any point, and a perpendicular upon it from S, falls in the given line AC. Now the curve which is generated by the perpetual intersections of these lines will also satisfy the problem; for each of the lines will be a tangent to it, and therefore perpendiculars from S upon its tangents will intersect them in the various points of AC. To get the equation to this curve we must, according to the usual method, differentiate with respect to the parameter C, which gives

$$0 = x + 2Ca,$$

and eliminate C between this, and the eqnation

$$y = Cx + a(1 + C^2),$$

which gives

$$y = -\frac{x^2}{4a} + a, \quad \text{or } 4a(a-y) = x^2,$$

the equation to a parabola, vertex A, focus S, of which curve it is a well-known property that the perpendicular from the focus intersects the tangent at any point, in the line touching the parabola at its vertex.

This result being obtained by exactly the same process as the singular solution was obtained, of course coincides with it; hence it appears that the singular solution belongs to the curve which touches the family of curves resulting from the complete integral by making the arbitrary constant assume all possible values. The conclusions arrived at in this particular instance, we shall now shew to hold generally.

39. Having given the complete integral of a differential equation, to find its singular solution.

Let $f\left(x, y, \frac{dy}{dx}\right) = 0$ be a proposed differential equation, and suppose it to result from the elimination of the arbitrary constant c, between the equation $F(x, y, c) = 0$, and its immediately derived equation

$$M + N\frac{dy}{dx} = 0.$$

Now change c into c' any function of x and y, then our equation becomes $F(x, y, c') = 0$; and its immediately derived equation is

$$M' + N'\frac{dy}{dx} + C'\frac{dc'}{dx} = 0 \quad \text{..................(1)},$$

c' entering into M' and N' just as c did into M and N, and $\frac{dc'}{dx}$ being formed in the usual way for a function of two variables x and y, the latter of which is dependent on the former. Now C' is the differential coefficient of $F(x, y, c')$ with respect to c', regarding x and y as constant, and will therefore usually involve x, y, and c'; and if put equal to zero, will give such a value for c' as makes the last term of equation (1) disappear;

and then the elimination of c' must evidently produce the proposed equation $f\left(x, y, \frac{dy}{dx}\right) = 0$. Let this value of c' be substituted in $F(x, y, c') = 0$; then this equation is changed into $\phi(x, y) = 0$, and furnishes a relation between x and y which satisfies the equation $f\left(x, y, \frac{dy}{dx}\right) = 0$, but contains no arbitrary constant, and is not deducible from the complete integral by giving a particular value to the constant; since it results from the complete integral by substituting for c a variable value deduced from the equation $\frac{d}{dc} F(x, y, c) = 0$. Consequently, the relation $\phi(x, y) = 0$ is the singular solution required.

40. To explain the geometrical signification of the singular solution of a differential equation.

Let $F(x, y, c) = 0$ (1) be the complete integral of a differential equation between two variables; if we differentiate it with regard to c, we have $\frac{d}{dc} F(x, y, c) = 0$ (2); and if between these equations we eliminate c, we get $\phi(x, y) = 0$ (3) where c does not appear, and which is a singular solution of the differential equation of which (1) is the complete primitive. Suppose equation (1) to be the equation to a system of curves, in which the position and dimensions of any particular curve is defined by a particular value of the parameter c; also let equation (3) be the equation to a curve referred to the same co-ordinate axes. Now equations (1) and (2), when c receives a certain value, are satisfied by the same values of x and y. Hence from the manner of its formation, equation (3) is satisfied by the same values; or the curves which are represented by (3) and (1), with a particular value of c, have a common point. But equations (3) and (1), being each a solution of the same differential equation, furnish the same value of $\frac{dy}{dx}$ for the same values of x and y; consequently the curves touch one another at their common point. The same thing happens for every one of the system of curves which equation (1) represents. Therefore the curve represented by the singular solution touches in a point every curve represented

by the complete primitive. This is the geometrical interpretation of the singular solution of a differential equation of the first order.

41. Having given a solution of a differential equation, to find whether it is included in the complete integral or not.

Let $\frac{dy}{dx} = f(x, y)$ be the proposed differential equation, and $y = F(x, c)$ its complete integral, c being the arbitrary constant; and when $c = c'$, let this become $y = u$, u containing no arbitrary constant; then $y = u$ is a particular integral of the proposed.

Hence since $F(x, c) - u$ becomes zero when $c = c'$, we have

$$F(x, c) - u = (c - c')^m . z = az \text{ suppose,}$$

z being a function of x and c, which is neither infinite nor zero when $c = c'$, or when $a = 0$; and m expressing the highest power of $c - c'$ which enters into every term of $F(x, c) - u$. Consequently the complete integral becomes

$$y = u + az,$$

which being substituted in the proposed equation,

$$\frac{dy}{dx} = f(x, y),$$

$$\text{gives } \frac{dy}{dx} + a\frac{dz}{dx} = f(x, u + az) \quad (1).$$

Now since z is neither infinite nor zero when $a = 0$, we may expand it in a series of ascending powers of a, in the form

$$z = K + Aa^\alpha + Ba^\beta + \&c.,$$

α, β, &c. being increasing and positive, and K, A, B, &c. functions of x;

$$\therefore \frac{du}{dx} + a\frac{dz}{dx} = \frac{du}{dx} + \frac{dK}{dx}a + \frac{dA}{dx}a^{\alpha+1} + \&c.$$

Again, the development of $f(x, u + az)$ will be of the form

$$f(x, u + az) = f(x, u) + M(az)^m + N(az)^n + \&c.,$$

m, n, &c. being increasing and positive. Hence, by substitution in equation (1), observing that $\frac{du}{dx} = f(x, u)$, we get

$$\frac{dK}{dx} \cdot a + \frac{dA}{dx} \cdot a^{a+1} + \&c.$$

$$= Ma^m (K + Aa^a + \&c.)^m + Na^n (K + Aa^a + \&c.)^n + \&c.$$

Now unless this equation is identical, $y = u$ cannot result from $y = F(x, c)$ by changing c into c'; and therefore $y = u$ cannot be a particular integral of the proposed; and consequently if it satisfies the proposed, it must be a singular solution. Now the indices m, n, &c., are known, for they result from writing $u + az$ for y in $f(x, y)$ and expanding according to powers of az; and we must endeavour to determine α, β, &c. so that the two members of the equation may be identical. If m be > 1, this can easily be effected; for we must have $\frac{dK}{dx} = 0$, or $K =$ a constant; $a + 1 = m$, and $\frac{dA}{dx} = MK^m$; and so on for the other terms. Consequently it will be possible to make the two members identical, and $y = u$ will be a particular integral. In the same way the identity may be established if $m = 1$. But if $m < 1$, there is no term on the first side corresponding to $MK^m a^m$; and since K cannot be equal to zero, it is impossible to satisfy the identity; and therefore $y = u$ is a singular solution. Hence to discover whether a given solution, $y = u$, of a differential equation

$$\frac{dy}{dx} = f(x, y),$$

is a singular solution or not; we must write $u + h$ for y in the value of $\frac{dy}{dx}$, and if the expansion in ascending powers of h involve a power of h, whose index is < 1, the solution in question is a singular solution; otherwise it is a particular integral.

42. To deduce the singular solutions from the differential equation, without knowing its complete primitive.

Let $y = u$ be a singular solution of the equation

$$\frac{dy}{dx} = f(x, y);$$

then by the preceding article, substituting $u + h$ for y, we get

$$f(x, u + h) = f(x, u) + Mh^m + Nh^n + \&c.$$

where m, n, &c. are proper fractions;

$$\therefore \frac{d}{du}f(x, u+h) = \frac{d}{dh}f(x, u+h) = mMh^{m-1} + nNh^{n-1} + \&c.;$$

consequently, when $h=0$, $\frac{d}{du}f(x, u) = \infty$.

But $\frac{d}{du}f(x, u)$ is what $\frac{d}{dy}f(x, y)$ becomes when $y=u$; and therefore, conversely, every value u of y, which satisfies $\frac{dy}{dx} = f(x, y)$, and makes $\frac{d}{dy}f(x, y) = \infty$, is a singular solution of the equation

$$\frac{dy}{dx} = f(x, y).$$

43. It is not essential to give the equation the explicit form $\frac{dy}{dx} = f(x, y)$. For let $\frac{dy}{dx} = p$, and let $V=0$ be the given relation between x, y and p; then we may regard p as a function of x and y determined by the equation $V=0$; hence forming the differential coefficients $\frac{dV}{dy}$, $\frac{dV}{dp}$ as if y and p were independent of one another, we get

$$\frac{dV}{dy} + \frac{dV}{dp}\frac{dp}{dy} = 0, \quad \text{or} \quad \frac{dp}{dy} = -\frac{dV}{dy} \div \frac{dV}{dp}.$$

Hence the condition $\frac{d}{dy}f(x, y) = \infty$ is equivalent to $\frac{dV}{dp} = 0$, provided $\frac{dV}{dy}$ remains finite; and therefore the singular solutions of the equation $V = F(x, y, p) = 0$, are determined by eliminating p between $V=0$ and $\frac{dV}{dp} = 0$, provided always that these solutions satisfy the proposed equation and do not make $\frac{dV}{dy} = 0$. And conversely if the solution of an equation be given, and we deduce from it the value of $\frac{dy}{dx} = p$, then if this value make $\frac{dV}{dp}$ vanish without making at the same time $\frac{dV}{dy} = 0$, it is

a singular solution; otherwise it is a particular integral. It is evident that if we consider y as the independent variable, and put the equation under the form

$$V = F\left(x, y, \frac{dx}{dy}\right) = 0,$$

the same reasonings would shew that singular solutions may be obtained by eliminating $p' = \frac{dx}{dy}$, between $V = 0$ and $\frac{dV}{dp'} = 0$, provided that these solutions do not at the same time make $\frac{dV}{dx} = 0$.

Ex. 1. To find the singular solution of

$$y - x\frac{dy}{dx} + x - \frac{y}{\frac{dy}{dx}} = a.$$

Here $\frac{dV}{dp} = -x + \frac{y}{p^2} = 0$; $\therefore\ p^2 = \frac{y}{x}$, and the proposed becomes

$$(x + y - a)\,p = y + xp^2, \quad \text{or} \quad (x + y - a)\,p = 2y,$$

$$\text{or} \quad (x + y - a)^2 = 4xy,$$

which may be reduced to the form $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

2. $\left(y - x\frac{dy}{dx}\right)\left(x - \frac{y}{\frac{dy}{dx}}\right) = a^2.\quad 4xy = a^2.$

3. $\left(y - x\frac{dy}{dx}\right)^2 + \left(x - \frac{y}{\frac{dy}{dx}}\right)^2 = a^2.\quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

4. $\left(y - x\frac{dy}{dx}\right)\left\{y - (x - 2a)\frac{dy}{dx}\right\} = b^2,\ y^2 = \frac{b^2}{a^2}(2ax - x^2).$

The three former examples determine respectively the curves which have the properties that $OT + OT'$ is constant, that the area of the triangle TOT' is constant, and that TT' is constant, TT' being the tangent at any point meeting the axes Ox, Oy in T and T', (fig. 5); the last determines a curve such that the

product of the portions of two fixed parallel straight lines intercepted between the tangent at any point and the axis of x, shall be invariable.

5. $$y + \left\{x - \tan^{-1}\left(\frac{dy}{dx}\right)\right\} \frac{dy}{dx} - 1 = 0.$$

$$\text{Here } \frac{dV}{dp} = x - \tan^{-1} p - \frac{p}{1+p^2} = 0\,;$$

$$\therefore\ x = \tan^{-1} p + \frac{p}{1+p^2}\,;$$

$$y = \frac{1}{1+p^2}, \quad \therefore\ x = \cos^{-1}\sqrt{y} + \sqrt{y - y^2},$$

which represents a cycloid whose base coincides with the axis of x, the origin being in the centre of the base. This is the solution of the problem to find a curve always touched by the same diameter of a circle rolling along a straight line.

6. $$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+y^2-a^2} - y}\,;$$

$$\therefore\ \frac{dp}{dy} = \frac{-x}{(\sqrt{x^2+y^2-a^2}-y)^2}\left(\frac{y}{\sqrt{x^2+y^2-a^2}} - 1\right);$$

$\therefore$ the relation $x^2+y^2-a^2=0$ makes $\frac{dp}{dy}$ infinite, and satisfies the proposed equation; it is consequently a singular solution of the proposed. But if we suppose the solution given, we may find whether it is comprised in the complete integral or not, by Art. 41. For we have $y = \sqrt{a^2-x^2}$; therefore substituting $\sqrt{a^2-x^2}+h$ for y in the value of $\frac{dy}{dx}$, we get

$$\frac{x}{\sqrt{2h\sqrt{a^2-x^2}+h^2} - \sqrt{a^2-x^2} - h} = \frac{-x}{\sqrt{a^2-x^2}}\left\{1 + \frac{\sqrt{2h}}{(a^2-x^2)^{\frac{1}{4}}} - \&c.\right\},$$

when developed according to powers of h; and as the index of h is a proper fraction, $x^2+y^2-a^2=0$ cannot be comprised in the complete integral, and is therefore a singular solution.

7. $(x-a)\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx}+a=0$, to find whether $y^2=4a(x-a)$, $y=x$, are particular integrals or singular solutions. We get

$$\frac{dV}{dp}=2(x-a)p-y, \quad \frac{dV}{dy}=-p.$$

Hence the solution $y=x$ which gives $p=1$, does not make $\frac{dV}{dp}$ vanish, and is therefore a particular integral. In fact the proposed equation being

$$y=(x-a)\frac{dy}{dx}+\frac{a}{\frac{dy}{dx}},$$

its complete integral is (Art. 34)

$$y=(x-a)\,C+\frac{a}{C};$$

and this becomes $y=x$, when $C=1$. But the solution

$$y^2=4a(x-a) \text{ gives } yp=2a;$$

and these values of y and p reduce $\frac{dV}{dp}$ to zero without making $\frac{dV}{dy}$ vanish; consequently $y^2=4a(x-a)$ is a singular solution.

8. $y^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}+ax+by=0.$

$$\frac{dV}{dp}=2y(py+x), \quad \frac{dV}{dy}=2p(py+x).$$

The value of p which makes $\frac{dV}{dp}$ vanish is $p=-\frac{x}{y}$, and the result of the elimination of p is $ax+by-x^2=0$; but as $\frac{dV}{dy}$ vanishes for the same value of p, this cannot be a singular solution. In fact, it does not satisfy the proposed equation, and is not a solution of any sort.

9. $\left(\frac{dy}{dx}\right)^2 - 2ay\frac{dy}{dx} + bx = 0,\quad \frac{dV}{dp} = 2p - 2ay,\quad \frac{dV}{dy} = -2ap$;
the value of p which makes $\frac{dV}{dp}$ vanish, is $p = ay$, and this value does not make $\frac{dV}{dy}$ vanish; but the result of the elimination of p, $a^2y^2 = bx$, as it does not satisfy the given equation, is not a singular solution.

44. Every factor proper to make a proposed differential equation integrable, is made infinite by the singular solution.

Let $\frac{dy}{dx} + f(x, y) = 0$ be the proposed equation, $F(x, y) = c$ its complete integral, and z the factor which makes it integrable, so that

$$z\left\{\frac{dy}{dx} + f(x, y)\right\} = \frac{d}{dx}F(x, y);$$

also, let $y = u$ be the singular solution; then this is not deducible from the complete integral, and therefore if u be written for y in $F(x, y)$, the result will not be constant; if therefore we substitute u for y, in the preceding equation, since the second member will have a finite value, and the factor $\frac{dy}{dx} + f(x, y)$ of the first member will be zero, the value of the other factor z corresponding to this substitution must be infinite.

This property will sometimes lead to the discovery of the factor which makes an equation integrable; as in the example

$$(a^2 - x^2)\frac{dy}{dx} + xy = a\sqrt{x^2 + y^2 - a^2},$$

a singular solution of which is $x^2 + y^2 - a^2 = 0$; if we try a factor of the form

$$(x^2 - a^2)^m\,(y^2 + x^2 - a^2)^n,$$

we arrive at $m = -1$, $n = -\frac{1}{2}$; and the factor which makes the proposed integrable is

$$(x^2 - a^2)^{-1}\,(x^2 + y^2 - a^2)^{-\frac{1}{2}}.$$

SECTION IV.

DIFFERENTIAL EQUATIONS OF THE SECOND ORDER, AND OF HIGHER ORDERS.

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45. EVERY differential equation of the n^{th} order admits of a primitive with n arbitrary constants.

Let $f(x, y, c_1, c_2, \dots c_n) = 0$ be an equation between the variables x and y, containing n constants $c_1, c_2, \dots c_n$. Let the first n derived equations be

$$f_1\left(x, y, \frac{dy}{dx}, c_1, \dots c_n\right) = 0,$$

$$f_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, c_1, \dots c_n\right) = 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f_n\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}, c_1, \dots c_n\right) = 0.$$

Between these n equations and the original, the n constants may be eliminated, and the result will be

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0, \dots\dots\dots\dots (1)$$

a differential equation in which none of the constants enter.

Conversely, a differential equation of the n^{th} order being proposed, it must admit of a primitive containing n arbitrary constants, because that number of constants, and no more, can be eliminated in its formation. Hence every differential equation of the n^{th} order admits of a primitive containing n arbitrary constants.

46. Again, between the original equation and its first $n-1$ derived equations, $n-1$ of the constants may be eliminated,

and a differential equation of the $(n-1)^{\text{th}}$ order with one constant will result.

Every such differential equation, having the same primitive with equation (1), is a first integral of that equation; hence a differential equation of the n^{th} order has n first integrals, each a differential equation of the $(n-1)^{\text{th}}$ order, and containing one constant.

Also, between the original equation and the first r of its derived equations, r of the constants may be eliminated, and a differential equation of the r^{th} order containing $n-r$ constants will result, which is an integral of equation (1). Now r constants can be eliminated in a number of ways equal to the number of combinations of n things taken r together, or

$$\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}.$$

This then is the number of integrals which equation (1) has of the $(n-r)^{\text{th}}$ order, each a differential equation of the r^{th} order and containing $n-r$ constants.

47. Of the general equation of the second order

$$F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right)=0,$$

we shall first of all consider the following particular cases, in which $\frac{d^2y}{dx^2}$ is involved with only one, or two, of the other quantities x, y, $\frac{dy}{dx}$; and which admit of integration, or rather of reduction to forms of the first order.

I. $F\left(\frac{d^2y}{dx^2}, x\right)=0$. Let this by resolution give

$$\frac{d^2y}{dx^2}=f(x), \quad \therefore \frac{dy}{dx}=\int dx f(x)+C;$$

and integrating again and adding another constant, we obtain the complete integral. The same process applies to $\frac{d^ny}{dx^n}=f(x)$.

Also, if we have $F\left(\frac{d^{n+1}y}{dx^{n+1}}, \frac{d^n y}{dx^n}\right) = 0$, and put $\frac{d^n y}{dx^n} = u$, we get $F\left(\frac{du}{dx}, u\right) = 0$; and if this can be integrated, and gives $u = f(x)$, it is reduced to the case just noticed.

Ex. 1. $x^2 \frac{d^2y}{dx^2} = a, \quad \therefore \frac{d^2y}{dx^2} = \frac{a}{x^2}, \quad \frac{dy}{dx} = -\frac{a}{x} + C;$

$$\therefore y = a \log \frac{1}{x} + Cx + C'.$$

Ex. 2. $\frac{d^2y}{dx^2} \cdot \frac{d^3y}{dx^3} + x = 0, \quad \therefore \frac{d^2y}{dx^2} = \sqrt{c^2 - x^2};$

$$\therefore y = \frac{1}{6}(2c^2 + x^2)\sqrt{c^2 - x^2} + \frac{1}{2}c^2 x \sin^{-1}\frac{x}{c} + Cx + C'.$$

II. $F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}\right) = 0$. Let this by resolution give

$$\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}\right), \quad \text{or } \frac{dp}{dx} = f(p), \quad \text{putting } \frac{dy}{dx} = p;$$

$$\therefore \frac{dx}{dp} = \frac{1}{f(p)}, \quad x = \int dp \frac{1}{f(p)};$$

$$\text{and } \frac{dy}{dp} = \frac{dy}{dx} \cdot \frac{dx}{dp} = \frac{p}{f(p)}, \quad \therefore y = \int dp \frac{p}{f(p)};$$

p must be eliminated between these two equations.

Ex. $a\frac{d^2y}{dx^2} = \frac{dy}{dx}; \quad \therefore \frac{dp}{dx} = \frac{p}{a}; \quad \therefore \log p = \frac{x}{a} + C,$

$$\text{and } \frac{dy}{dp} = \frac{dy}{dx} \cdot \frac{dx}{dp} = p \cdot \frac{a}{p} = a; \quad \therefore y = ap + C';$$

$$\therefore \frac{x}{a} + C = \log \frac{y - C'}{a}.$$

III. $F\left(\frac{d^2y}{dx^2}, y\right) = 0$. Let this give $\frac{d^2y}{dx^2} = f(y)$;

$$\therefore\ 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx}, \text{ and } \left(\frac{dy}{dx}\right)^2 = C + 2\int dy f(y);$$

$$\therefore\ \frac{dy}{dx} = \sqrt{C + 2\int dy f(y)}, \text{ and } x = \int dy \frac{1}{\sqrt{C + 2\int dy f(y)}}.$$

Also $F\left(\dfrac{d^{n+2}y}{dx^{n+2}}, \dfrac{d^ny}{dx^n}\right) = 0$, putting $\dfrac{d^ny}{dx^n} = u$, becomes

$$F\left(\frac{d^2u}{dx^2}, u\right) = 0;$$

and this, treated as above, gives $x = \phi(u)$; and if $x = \phi(u)$ can be solved with respect to u, it is brought under Case I.

Ex. 1. $a^2\dfrac{d^2y}{dx^2} + y = 0;\quad \therefore\ 2a^2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} + 2y\dfrac{dy}{dx} = 0;$

$$\therefore\ a^2\left(\frac{dy}{dx}\right)^2 + y^2 = C;\quad \therefore\ \frac{a}{\sqrt{C - y^2}}\cdot\frac{dy}{dx} = 1,$$

$$\therefore\ a\sin^{-1}\frac{y}{\sqrt{C}} = x + C'.$$

Ex. 2. $ny^3\dfrac{d^2y}{dx^2} - 1 = 0;\quad \dfrac{1}{n}(x - c')^2 = \dfrac{1}{c}y^2 - \dfrac{1}{c^2}.$

This is the solution of the problem, in which a curve concave towards the axis of x is sought whose radius of curvature shall vary as the cube of its normal: for this requires that

$$(1 + p^2)^{\frac{3}{2}} \div \frac{dp}{dx} = ny^3(1 + p^2)^{\frac{3}{2}}.$$

Ex. 3. $\sqrt{ay}\dfrac{d^2y}{dx^2} = 1.$

$$\frac{x}{\sqrt[4]{a}} = \frac{2}{3}(\sqrt{y} + C)^{\frac{3}{2}} - 2C\sqrt{\sqrt{y} + C} + C'.$$

IV. $F\left(\dfrac{d^2y}{dx^2}, \dfrac{dy}{dx}, x\right) = 0.$

This becomes of the first order in p and x, by putting p for $\dfrac{dy}{dx}$ and $\dfrac{dp}{dx}$ for $\dfrac{d^2y}{dx^2}$; let its integral be $\phi(x, p, C) = 0$.

If this by resolution give p or $\frac{dy}{dx} = f(x)$, then $y = \int dx f(x)$; among other cases, this will happen when the proposed is of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} = Q$; for the latter can be solved as a linear equation of the first order.

If it gives $x = f(p)$, then

$$y = \int dx p = xp - \int dx\, x \frac{dp}{dx} = xp - \int dp f(p)\,;$$

and p must be eliminated between these equations.

Ex. 1. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = 0$; $\quad \therefore\ x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = 0$;

$$\therefore\ x\frac{dy}{dx} = C; \quad \therefore\ y = C\log x + C'.$$

Ex. 2. $(1 + x^2)\dfrac{d^2y}{dx^2} + 1 + \left(\dfrac{dy}{dx}\right)^2 = 0$;

$$\therefore\ \frac{1}{1+p^2}\frac{dp}{dx} + \frac{1}{1+x^2} = 0, \quad \tan^{-1}p + \tan^{-1}x = \tan^{-1}c\,;$$

$$\therefore\ p = \frac{c - x}{1 + cx} = \frac{1}{c}\left(\frac{1 + c^2}{1 + cx} - 1\right);$$

$$\therefore\ y = \frac{1 + c^2}{c^2}\log(1 + cx) - \frac{x}{c} + C'.$$

Ex. 3. $x(a + bx)\dfrac{d^2y}{dx^2} + (c + ex)\dfrac{dy}{dx} = 0$;

$$\therefore\ \frac{1}{p}\frac{dp}{dx} = -\frac{c + ex}{x(a + bx)} = -\frac{cx^{-2}}{ax^{-1} + b} - \frac{e}{a + bx};$$

$$\therefore\ p = c'x^{-\frac{c}{a}}(a + bx)^{\frac{c}{a} - \frac{e}{b}} \text{ and } y = C + c'\int dx\, x^{-\frac{c}{a}}(a + bx)^{\frac{c}{a} - \frac{e}{b}}.$$

Ex. 4. $\dfrac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = f(x)$; $\quad \therefore\ \dfrac{1}{(1 + p^2)^{\frac{3}{2}}}\dfrac{dp}{dx} = \dfrac{1}{f(x)}$;

$$\therefore \frac{p}{\sqrt{1+p^2}} = \int dx \frac{1}{f(x)} = X, \text{ suppose};$$

$$\therefore p = \frac{X}{\sqrt{1-X^2}}, \text{ and } y = \int dx \frac{X}{\sqrt{1-X^2}}.$$

This is the solution of the inverse problem of the radius of curvature, in which it is required to find a curve whose radius of curvature shall be a given function of the abscissa.

Ex. 5. $\frac{d^2y}{dx^2} + (e^x - 1)\frac{dy}{dx} = e^{2x}, \quad y = e^x + Ce^{-e^x} + C'.$

V. $F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0.$

Putting $\frac{dy}{dx} = p$, we get

$$\frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p\frac{dp}{dy};$$

and the substitution of $p\frac{dp}{dy}$ for $\frac{d^2y}{dx^2}$, and of p for $\frac{dy}{dx}$, will make the proposed of the first order in p and y; let its integral be

$$\phi(p, y, C) = 0.$$

If this by resolution give $p = \frac{dy}{dx} = f(y)$, then

$$x = \int dy \frac{1}{f(y)}.$$

If it gives $y = f(p)$, then

$$x = \int dy \frac{1}{p} = \frac{y}{p} + \int dy \frac{y}{p^2}\frac{dp}{dy} = \frac{y}{p} + \int dp \frac{f(p)}{p^2},$$

and p must be eliminated between these equations.

Ex. 1. $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0, \quad \text{or} \quad \frac{d}{dx}\left(y\frac{dy}{dx}\right) = 0;$

$$\therefore y\frac{dy}{dx} = C; \quad \therefore y^2 = 2Cx + C'.$$

Ex. 2. $\quad y\dfrac{d^2y}{dx^2}+n\left(\dfrac{dy}{dx}\right)^2+n=0;$

$$\therefore\ yp\frac{dp}{dy}+np^2+n=0;$$

$$\therefore\ \tfrac{1}{2}\log\left(1+p^2\right)+n\log y=\tfrac{1}{2}\log C;$$

$$\therefore\ \left(1+p^2\right)y^{2n}=C;$$

$$\therefore\ p=\sqrt{Cy^{-2n}-1},\quad\text{and}\quad x=\int dy\,\frac{y^n}{\sqrt{C-y^{2n}}}.$$

This is the solution of the problem in which it is required to find a curve whose radius of curvature shall vary as its normal; for this condition gives

$$y\sqrt{1+p^2}=\frac{n\left(1+p^2\right)^{\frac{3}{2}}}{\pm\dfrac{dp}{dx}},\quad\text{or}\quad y\frac{dp}{dx}+(\mp n)\left(1+p^2\right)=0,$$

$-$ or $+$ according as the curve is convex or concave to the axis of x. If $n=1$ the curve is a circle, if $n=2$ a cycloid, if $n=-1$ a common catenary.

Ex. 3. $\quad 1+\left(\dfrac{dy}{dx}\right)^2-2y\dfrac{d^2y}{dx^2}=3ay\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}.$

This gives $\qquad \dfrac{dx}{dy}=\dfrac{ay^{\frac{3}{2}}+C}{\sqrt{y-\left(ay^{\frac{3}{2}}+C\right)^2}}.$

Ex. 4. $\quad y^2\dfrac{d^2y}{dx^2}+\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=0,$ gives

$$\frac{dx}{dy}=\frac{y}{\sqrt{(Cy+1)^2-y^2}}.$$

VI. $\quad F\left(\dfrac{d^2y}{dx^2},\ y,\ x\right)=0.$

As there is no substitution by which this can be generally reduced to an equation of the first order between two variables, the artifice to be employed in any case will depend upon the

nature of the example proposed. Among other substitutions for $\frac{d^2y}{dx^2}$, the two following may be noticed,

$$x\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{x^2\frac{d}{dx}\left(\frac{y}{x}\right)\right\}, \text{ and } x\frac{d^2y}{dx^2} = \frac{d^2(xy)}{dx^2} - 2\frac{dy}{dx}.$$

Ex. 1. $x^2\frac{d^2y}{dx^2} = 2y$;

$$\therefore \frac{d^2(xy)}{dx^2} - 2\frac{dy}{dx} = \frac{2y}{x}, \text{ or } \frac{d^2(xy)}{dx^2} = \frac{2}{x}\frac{d(xy)}{dx};$$

$$\therefore \frac{d}{dx}(xy) = Cx^2; \text{ and } y = \frac{Cx^2}{3} + \frac{C'}{x}.$$

Ex. 2. $(x^2+y^2)^2\frac{d^2y}{dx^2} + a^2y = 0$, let $\frac{y}{x} = z$, then

$$\frac{d}{dx}\left(x^2\frac{dz}{dx}\right) + \frac{a^2}{x^2}\frac{z}{(1+z^2)^2} = 0;$$

$$\therefore x^2\frac{dz}{dx}\cdot\frac{d}{dx}\left(x^2\frac{dz}{dx}\right) + \frac{a^2z}{(1+z^2)^2}\frac{dz}{dx} = 0;$$

$$\therefore \left(x^2\frac{dz}{dx}\right)^2 - \frac{a^2}{1+z^2} = C,$$

$$\text{or } x^2\frac{dz}{dx} = \sqrt{C + \frac{a^2}{1+z^2}},$$

$$\text{or } -\frac{1}{x} = \int dz\,\frac{\sqrt{1+z^2}}{\sqrt{a^2 + C(1+z^2)}}.$$

Ex. 3. $\frac{d^2y}{dx^2} = ax + by$;

$$\text{or } \frac{d^2}{dx^2}(ax+by) = b(ax+by),$$

which becomes $\frac{d^2z}{dx^2} = bz$, putting $ax + by = z$, and so falls under Case III.

48. Of the general equation of the second order involving x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, the following particular cases may be noticed, in which P denotes a function of x.

1. $P\frac{d^2y}{dx^2}+\frac{1}{2}\frac{dP}{dx}\cdot\frac{dy}{dx}+cy=0$;

this when multiplied by $2\frac{dy}{dx}$ may be written

$$\frac{d}{dx}\left\{P\left(\frac{dy}{dx}\right)^2\right\}+c\frac{d}{dx}y^2=0;$$

$$\therefore\ P\left(\frac{dy}{dx}\right)^2+cy^2=C,\ \text{ or }\ \frac{1}{\sqrt{C-cy^2}}\frac{dy}{dx}=\frac{1}{\sqrt{P}}.$$

Thus if $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+q^2y=0$;

$$\therefore\ \frac{1}{\sqrt{C-q^2y^2}}\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}},\ \text{ or }\ \frac{1}{q}\sin^{-1}\frac{qy}{\sqrt{C}}=\sin^{-1}x+C',$$

which may be put under the form

$$y=C_1\cos(q\sin^{-1}x)+C_2\sin(q\sin^{-1}x).$$

Similarly, $(ax+bx^2)\frac{d^2y}{dx^2}+\left(\frac{1}{2}a+bx\right)\frac{dy}{dx}+cy=0$,

leads to $\frac{1}{\sqrt{C-cy^2}}\frac{dy}{dx}=\frac{1}{\sqrt{ax+bx^2}}$.

2. $\frac{d^2y}{dx^2}+P\left(\frac{dy}{dx}-\frac{y}{x}\right)=0$;

$$\therefore\ \frac{d}{dx}\left(x\frac{dy}{dx}-y\right)+P\left(x\frac{dy}{dx}-y\right)=0;$$

$$\therefore\ x\frac{dy}{dx}-y=Ce^{-\int dxP}\ \text{ or }\ \frac{d}{dx}\left(\frac{y}{x}\right)=\frac{C}{x^2}e^{-\int dxP}.$$

$$\therefore\ y=Cx\int dx\,.\,\frac{1}{x^2}e^{-\int dxP}.$$

48*. When the equation

$$F\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}\right)=0, \quad \text{or } F\left(x,\ y,\ p,\ \frac{dp}{dx}\right)=0,$$

is homogeneous, reckoning the dimensions of p and $\frac{dp}{dx}$ to be 0 and -1 respectively; it may be reduced to an equation of the first order by putting

$$y=xz, \quad \text{and } \frac{dp}{dx}=\frac{q}{x}.$$

For each term, if r denote its dimensions, will consist of some function of p multiplied by a factor of the form

$$\left(\frac{y}{x}\right)^m x^{n+r}\left(\frac{dp}{dx}\right)^n,$$

m and n being any numbers from 0 to ∞; therefore upon making the substitutions stated above, every term will be divisible by x^r, and the equation will assume the form

$$F(p,\ q,\ z)=0;$$

and if this can be solved relative to q, we shall have

$$q=\phi(z,\ p); \quad \text{but } q=x\frac{dp}{dx}=\frac{(p-z)}{\frac{dz}{dx}}\frac{dp}{dx}=(p-z)\frac{dp}{dz};$$

$$\therefore\ \phi(z,\ p)+(z-p)\frac{dp}{dz}=0;$$

let this give

$$p=\psi(z), \quad \text{then } \psi(z)=z+x\frac{dz}{dx},$$

which will give the required integral $z=\frac{y}{x}=f(x)$.

Ex. $$n\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=\frac{d^2y}{dx^2}\sqrt{x^2+y^2};$$

putting $\frac{dy}{dx}=p$, $y=xz$, $\frac{dp}{dx}=\frac{q}{x}$, we find

$$n(1+p^2)^{\frac{3}{2}}=q\sqrt{1+z^2};$$

$$\therefore\ q = \frac{n(1+p^2)^{\frac{3}{2}}}{\sqrt{1+z^2}} = (p-z)\frac{dp}{dz},$$

which is the same as Ex. 5, Art. 18.

Linear Equations of the Second and Higher Orders.

49. The linear equation of the n^{th} order is

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + p_{n-1}\frac{dy}{dx} + p_n y = X,$$

all the coefficients being functions of x, and each term of the first member involving either y, or one of its differential coefficients, in the first power. The first step towards the integration of this equation is the establishment of the following theorem.

If there be n particular values $u_1, u_2, u_3 \dots u_n$, functions of x, which, when substituted for y, satisfy the equation

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + p_n y = 0,$$

its complete integral is

$$y = a_1 u_1 + a_2 u_2 + a_3 u_3 + \dots + a_n u_n,$$

$a_1, a_2, \dots a_n$ being arbitrary constants.

For let this value of y be substituted in the expression

$$\frac{d^n y}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + p_n y,$$

and it becomes

$$\frac{d^n}{dx^n}(a_1u_1 + a_2u_2 + \dots + a_nu_n)$$

$$+ p_1\frac{d^{n-1}}{dx^{n-1}}(a_1u_1 + a_2u_2 + \dots + a_nu_n) + \dots + p_n(a_1u_1 + a_2u_2 + \dots + a_nu_n),$$

or, collecting the terms multiplied by the factors $a_1, a_2, \dots a_n$,

$$a_1\left(\frac{d^n u_1}{dx^n} + p_1\frac{d^{n-1}u_1}{dx^{n-1}} + \dots + p_n u_1\right)$$

$$+ a_2\left(\frac{d^n u_2}{dx^n} + p_1\frac{d^{n-1}u_2}{dx^{n-1}} + \dots + p_n u_2\right) + \dots$$

$$+ a_n\left(\frac{d^n u_n}{dx^n} + p_1\frac{d^{n-1}u_n}{dx^{n-1}} + \dots p_n u_n\right);$$

Now, since $u_1, u_2, \dots u_n$ satisfy the equation, each of the quantities within brackets is equal to zero, and therefore the whole is identically zero; and therefore the assumed value of y satisfies the equation; and it contains n arbitrary constants, consequently it is the complete integral of the equation.

Thus the equation $\frac{d^2y}{dx^2} - n^2y = 0$ is satisfied by $y = e^{nx}$, and since it is not altered by changing the sign of n, it is also satisfied by $y = e^{-nx}$; $\therefore\ y = a_1e^{nx} + a_2e^{-nx}$ is the complete integral.

Linear Equations with constant Coefficients.

50. To integrate the equation

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \dots + p_ny = q,$$

all the coefficients and q being constant.

First write $y + \frac{q}{p_n}$ instead of y; and the equation becomes

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + p_ny = 0,$$

which shews that we may treat the proposed equation as if it had no term q independent of y, provided we divide that term by the coefficient of y and add the quotient to the value of y obtained on the supposition that $q = 0$.

Let $y = e^{mx}$, $\therefore\ e^{mx}(m^n + p_1m^{n-1} + \dots + p_n)$ is the value of the first member; now this will vanish if m be any root of the equation

$$m^n + p_1m^{n-1} + p_2m^{n-2} + \dots + p_n = 0, \text{ or } M = f(m) = 0,$$

which is called the auxiliary equation of the proposed linear equation.

Hence the n real or imaginary roots of this equation, $m_1, m_2, m_3 \dots m_n$, provided they be all unequal, will give n different particular values of y, $e^{m_1x}, e^{m_2x}, \dots e^{m_nx}$, which satisfy the proposed equation; and therefore its complete integral is

$$y = a_1e^{m_1x} + a_2e^{m_2x} + \dots + a_ne^{m_nx}.$$

51. But if any of the roots are equal to one another, as, for instance, $m_1 = m_2$, the value of y becomes

$$y = (a_1 + a_2)\, e^{m_1x} + a_3 e^{m_3x} + \dots\, a_n e^{m_nx},$$

which contains only $n-1$ arbitrary constants (because $a_1 + a_2$ can be reckoned only as a single constant), and therefore cannot be the complete integral of the proposed equation. In this case, in order to discover the complete integral, first suppose the two roots m_1, m_2, to be only very nearly equal to one another, so that $m_2 = m_1 + h$, where h is a very small known quantity; then the part of the value of y corresponding to these roots is

$$a_1 e^{m_1x} + a_2 e^{m_2x} = e^{m_1x}\,(a_1 + a_2 e^{hx}) = e^{m_1x}\left(a_1 + a_2 + a_2 \frac{hx}{1} + a_2 \frac{h^2x^2}{1.2} + \&c.\right)$$
$$= e^{m_1x}\,(c_1 + c_2 x + \tfrac{1}{2} c_2 h x^2 + \tfrac{1}{6} c_2 h^2 x^3 + \&c.),$$

replacing the constants $a_1 + a_2$ and $a_2 h$, by c_1, c_2 respectively. Now let $h = 0$, then this becomes $e^{m_1x}(c_1 + c_2 x)$; and the complete integral consequently is

$$y = (c_1 + c_2 x)\, e^{m_1x} + a_3 e^{m_3x} + \dots + a_n e^{m_nx}.$$

52. Generally, if we suppose r roots of the auxiliary equation to be nearly equal to one another, and therefore to be represented by

$$m_1 + h_1, \quad m_1 + h_2, \dots\, m_1 + h_r,$$

where $h_1, h_2, \dots h_r$ are very small quantities, the complete integral takes the form

$$y = e^{m_1x}\,(a_1 e^{h_1x} + a_2 e^{h_2x} + \dots + a_r e^{h_rx}) + a_{r+1} e^{m_{r+1}x} + \&c.,$$

or, expanding e^{h_1x}, e^{h_2x}, &c.,

$$y = e^{m_1x}\left\{\Sigma(a) + \dots + \frac{\Sigma(ah^{r-1})}{\lfloor r-1}\, x^{r-1} + \frac{\Sigma(ah^r)}{\lfloor r}\, x^r + \dots\right\} + a_{r+1} e^{xm_{r+1}} + \&c.,$$

or, replacing the constants

$$\Sigma(a), \;\; \Sigma(ah), \dots \frac{\Sigma(ah^{r-1})}{\lfloor r-1},$$

by $c_1, c_2 \dots c_r$,

$$y = e^{m_1x}(c_1 + c_2x + \dots + c_rx^{r-1} + \text{terms multiplied by } h_1, h_2, \text{\&c.})$$
$$+ a_{r+1}e^{xm_{r+1}} + \text{\&c.}$$

Now let $h_1 = 0 = h_2 = \dots = h_r$, in which case the auxiliary equation has r roots each $= m_1$; then the solution becomes

$$y = e^{m_1x}(c_1 + c_2x + c_3x^2 + \dots + c_rx^{r-1}) + a_{r+1}e^{m_{r+1}x} + \dots + a_ne^{m_nx},$$

which contains n arbitrary constants, and is consequently the complete integral.

53. Of the correctness of the above modification for the case of equal roots, we may assure ourselves by the following reverse process. Let $y = e^{mx}u$, then since by Leibnitz's theorem

$$\frac{d^n(uv)}{dx^n} = \frac{d^nv}{dx^n}u + n\frac{d^{n-1}v}{dx^{n-1}}\cdot\frac{du}{dx} + \frac{n(n-1)}{1.2}\cdot\frac{d^{n-2}v}{dx^{n-2}}\cdot\frac{d^2u}{dx^2} + \text{\&c.},$$

$$\frac{d^n}{dx^n}(e^{mx}u) = e^{mx}\left\{m^n.u + nm^{n-1}.\frac{du}{dx} + \frac{n(n-1)}{1.2}m^{n-2}\frac{d^2u}{dx^2} + \text{\&c.}\right\}$$
$$= e^{mx}\left(m + \frac{d}{dx}\right)^n u,$$

separating the symbol of operation from that of quantity.

Hence the first member of the equation, or $f\left(\frac{d}{dx}\right)e^{mx}u$, becomes

$$e^{mx}\left(m + \frac{d}{dx}\right)^n u + p_1e^{mx}\left(m + \frac{d}{dx}\right)^{n-1}u$$
$$+ p_2e^{mx}\left(m + \frac{d}{dx}\right)^{n-2}u + \text{\&c.} + p_ne^{mx}u$$

$$= e^{mx}f\left(m + \frac{d}{dx}\right)u$$

$$= e^{mx}\left\{f(m)\,u + \frac{1}{1}f'(m)\frac{du}{dx} + \frac{1}{1.2}f''(m)\frac{d^2u}{dx^2} + \dots + \frac{1}{\underline{|n}}f^{(n)}(m)\frac{d^nu}{dx^n}\right\}.$$

Now let $m = m_1$, then since $f(m) = 0$ has r roots equal to m_1,

each of the quantities $f(m)$, $f'(m)$, ... $f^{(r-1)}(m)$ becomes equal to zero, and the first r terms vanish; and if

$$u = a_1 + a_2x + a_3x^2 + \dots + a_rx^{r-1},$$

then $\frac{d^ru}{dx^r}$, $\frac{d^{r+1}u}{dx^{r+1}}$,... $\frac{d^nu}{dx^n}$ likewise become zero, and all the remaining terms vanish. Hence for each group of r equal roots whether real or imaginary in the auxiliary equation, there will be in the value of y a term of the form

$$e^{m_1x}(a_1 + a_2x + a_3x^2 + \dots + a_rx^{r-1}).$$

In the treatment of the linear equation with constant coefficients

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \&c. + p_ny = \left\{\left(\frac{d}{dx}\right)^n + p_1\left(\frac{d}{dx}\right)^{n-1} + \dots + p_n\right\}y = X,$$

$$\text{or } f\left(\frac{d}{dx}\right)y = X,$$

great use will be made of the formulæ just obtained,

$$\left(\frac{d}{dx}+m\right)^n u = e^{-mx}\frac{d^n}{dx^n}(e^{mx}u), \quad f\left(\frac{d}{dx}+m\right)u = e^{-mx}f\left(\frac{d}{dx}\right)e^{mx}u.$$

When n is negative, since $\left(\frac{d}{dx}\right)^{-1}$ is equivalent to $\int dx$, the former leads to

$$\left(\frac{d}{dx}+m\right)^{-n} u = e^{-mx}\int^n dx^n\,(e^{mx}u).$$

54. Let $h \pm k\sqrt{-1}$ be a pair of imaginary roots in the auxiliary equation, then the corresponding terms in the value of y will be

$$Ce^{hx+kx\sqrt{-1}} + C'e^{hx-kx\sqrt{-1}},$$

$$\text{or } e^{hx}\{C(\cos kx + \sqrt{-1}\sin kx) + C'(\cos kx - \sqrt{-1}\sin kx)\}$$

$$= e^{hx}\{c_1\cos kx + c_2\sin kx\} \text{ changing the arbitrary constants,}$$

$$\text{or } = \sqrt{c_1^2+c_2^2}\,e^{hx}\left\{\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos kx + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin kx\right\}$$

$= \beta e^{hx} \cos(kx + \alpha)$, again changing the constants by putting

$$\sqrt{c_1^2 + c_2^2} = \beta, \text{ and } -\tan\alpha = \frac{c_2}{c_1}.$$

And if there should be r pairs of imaginary roots in the auxiliary equation, equal to $h \pm k\sqrt{-1}$, the corresponding part of the value of y will be

$$e^{hx}(a_1 + a_2 x + \dots + a_r x^{r-1})(\cos kx + \sqrt{-1}\sin kx)$$

$$+ e^{hx}(b_1 + b_2 x + \dots + b_r x^{r-1})(\cos kx - \sqrt{-1}\sin kx),$$

or, changing the arbitrary constants,

$$e^{hx}(\alpha_1 + \alpha_2 x + \dots + \alpha_r x^{r-1})\cos kx.$$

$$+ e^{hx}(\beta_1 + \beta_2 x + \dots + \beta_r x^{r-1})\sin kx.$$

55. We shall now give some examples of integrating linear equations with constant coefficients, by the elementary method just investigated.

Ex. 1. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = 0.$

Let $y = e^{mx}$, $\therefore\ m^2 e^{mx} + m e^{mx} - 2e^{mx} = 0$,

$$\text{or } m^2 + m - 2 = 0; \quad \therefore\ m = 1, \text{ or } -2;$$

$$\therefore\ y = C_1 e^x + C_2 e^{-2x}.$$

2. $a^2 \dfrac{d^4y}{dx^4} = \dfrac{d^2y}{dx^2}$, $\quad y = C_1 e^{\frac{x}{a}} + C_2 e^{-\frac{x}{a}} + C_3 x + C_4.$

3. $\dfrac{d^2y}{dx^2} + n^2 y = 0.$

Let $y = e^{mx}$; $\therefore\ m^2 + n^2 = 0$, $\quad$ or $m = \pm n\sqrt{-1}$;

$$\therefore\ y = C_1 e^{nx\sqrt{-1}} + C_2 e^{-nx\sqrt{-1}}$$

$$= C_1(\cos nx + \sqrt{-1}\sin nx) + C_2(\cos nx - \sqrt{-1}\sin nx)$$

$$= (C_1 + C_2)\cos nx + (C_1 - C_2)\sqrt{-1}\sin nx$$

$$= a_1 \cos nx + a_2 \sin nx, \text{ changing the constants,}$$

$$= \beta\,(\cos nx \cos \alpha - \sin nx \sin \alpha) = \beta \cos (nx + \alpha).$$

This equation, $\dfrac{d^2y}{dx^2} + n^2y = 0$, of which the solution is

$$y = a_1 \cos nx + a_2 \sin nx, \quad \text{or } y = \beta \cos (nx + \alpha),$$

is of very frequent occurrence. Hence also the solution of

4. $\dfrac{d^2y}{dx^2} + n^2y + ax + b = 0$, or

$$\frac{d^2}{dx^2}\left(y + \frac{ax}{n^2} + \frac{b}{n^2}\right) + n^2\left(y + \frac{ax}{n^2} + \frac{b}{n^2}\right) = 0, \text{ is}$$

$$y + \frac{ax}{n^2} + \frac{b}{n^2} = \beta \cos (nx + \alpha).$$

5. $\dfrac{d^2y}{dx^2} - 2m\dfrac{dy}{dx} + (m^2 + n^2)\,y = 0$;

$$y = e^{kx}, \quad \therefore\ k^2 - 2km + m^2 + n^2 = 0, \quad k = m \pm n\sqrt{-1};$$

$$\therefore\ y = e^{mx}\,(C \cos nx + C' \sin nx).$$

6. $\dfrac{d^2y}{dx^2} + 2m\dfrac{dy}{dx} + n^2y = 0$;

$$y = e^{-mx}\beta \cos (x\sqrt{n^2 - m^2} + \alpha), \quad (n > m).$$

7. $\dfrac{d^4y}{dx^4} - 2\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$;

$$y = e^{kx}, \quad (k^2 + 1)\,(k - 1)^2 = 0;$$

$$\therefore\ y = e^x\,(a_1 + a_2x) + \beta \cos (x + \alpha).$$

8. $\dfrac{d^4y}{dx^4} + 2n^2\dfrac{d^2y}{dx^2} + n^4y = q$;

$$y = \frac{q}{n^4} + (a + a_1x) \cos nx + (b + b_1x) \sin nx.$$

9. $$\frac{d^4y}{dx^4} + 4m\frac{d^3y}{dx^3} + 2(n^2+3m^2)\frac{d^2y}{dx^2} + 4m(m^2+n^2)\frac{dy}{dx}$$
$$+ (m^2+n^2)^2 y = q;$$
$$y = e^{kx},\ \{(k+m)^2+n^2\}^2 = 0;$$
$$\therefore\ y = \frac{q}{(m^2+n^2)^2} + e^{-mx}(a+a_1x)\cos nx + e^{-mx}(b+b_1x)\sin nx.$$

10. $$\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 4;$$
$$y - 4 = e^x(c_0 + c_1x + c_2x^2 + c_3x^3).$$

Integration of linear equations with constant coefficients by separation of symbols.

56. Before proceeding to the general case of linear equations, we shall consider the case of a linear equation with all its coefficients constant but having a function of x, X, for the term independent of y. This equation can be shortly treated by the method of separation of symbols: for if we separate the symbol of operation $\frac{d}{dx}$ from its subject y, the equation becomes

$$\left\{\left(\frac{d}{dx}\right)^n + p_1\left(\frac{d}{dx}\right)^{n-1} + \ldots + p_n\right\} y = X; \quad\ldots\ldots\ldots\ (1)$$

or if a_1, a_2, ... a_n be the n distinct roots of the auxiliary equation

$$f(z) = z^n + p_1z^{n-1} + \ldots + p_n = 0,$$
$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\ldots\left(\frac{d}{dx} - a_n\right) y = X;$$
$$\therefore\ y = \left\{A_1\left(\frac{d}{dx} - a_1\right)^{-1} + A_2\left(\frac{d}{dx} - a_2\right)^{-1} + \ldots + A_n\left(\frac{d}{dx} - a_n\right)^{-1}\right\} X,$$

A_1, A_2, &c., having the same values as in the resolution of

$$(z^n + p_1z^{n-1} + \ldots + p_n)^{-1}$$

into a similar series of terms, and being therefore known in terms of a_1, a_2, &c. respectively. (*Integral Calculus*, Art. 34.) But by Leibnitz's Theorem (Art. 53) we have,

$$\left(\frac{d}{dx} + m\right)^{-n} u = e^{-mx}(\int dx)^n e^{mx} u;$$

and in the present case

$$\left(\frac{d}{dx} - a_1\right)^{-1} X = e^{a_1 x} \int dx\, e^{-a_1 x} X;$$

$$\therefore\ y = A_1 e^{a_1 x} \int dx\, e^{-a_1 x} X + A_2 e^{a_2 x} \int dx\, e^{-a_2 x} X + \dots + A_n e^{a_n x} \int dx\, e^{-a_n x} X,$$

the arbitrary constants being reserved under the signs of integration; or if they be expressed, we must add to the above value of y,

$$C_1 e^{a_1 x} + C_2 e^{a_2 x} + \dots + C_n e^{a_n x},$$

which is called the complementary function, and is the value assumed by y when $X = 0$.

If we denote $\frac{d}{dz} f(z)$ by $f'(z)$, then since $A_1 = \frac{1}{f'(a_1)}$, the above result may be written

$$y = \Sigma \left\{ \frac{e^{ax}}{f'(a)} \int dx\, e^{-ax} X \right\}.$$

57. If the auxiliary equation have r roots equal to a_1, then from (1),

$$y = \left(\frac{d}{dx} - a_1\right)^{-r} \left(\frac{d}{dx} - a_{r+1}\right)^{-1} \dots \left(\frac{d}{dx} - a_n\right)^{-1} X,$$

and this function of $\frac{d}{dx}$ may be resolved into partial fractions (*Integral Calculus*, Art. 36), so that

$$y = \left\{ A_1 \left(\frac{d}{dx} - a_1\right)^{-1} + A_2 \left(\frac{d}{dx} - a_1\right)^{-2} + \dots \right.$$

$$\left. + A_r \left(\frac{d}{dx} - a_1\right)^{-r} + A_{r+1} \left(\frac{d}{dx} - a_{r+1}\right)^{-1} + \dots + A_n \left(\frac{d}{dx} - a_n\right)^{-1} \right\} X.$$

But $\left(\frac{d}{dx} - a_1\right)^{-m} X = e^{a_1 x} \left(\frac{d}{dx}\right)^{-m} e^{-a_1 x} X = e^{a_1 x} \int^m dx^m \left(e^{-a_1 x} X\right) +$

$$e^{a_1 x} \left(c_0 + c_1 x + \dots + c_{m-1} x^{m-1}\right), \quad c_0,\ c_1\ \text{\&c.},$$

being the constants introduced after each integration,

$$\therefore\ y = A_1 e^{a_1 x}\int dx\,(e^{-a_1 x}X) + A_2 e^{a_1 x}\int^2 dx^2\,(e^{-a_1 x}X) + \dots$$
$$+ A_r e^{a_1 x}\int^r dx^r\,(e^{-a_1 x}X).$$
$$+ A_{r+1} e^{x a_{r+1}}\int dx\,(e^{-x a_{r+1}}X) + \dots + A_n e^{x a_n}\int dx\,(e^{-x a_n}X)$$
$$+ (C_0 + C_1 x + \dots + C_{r-1}x^{r-1})\,e^{a_1 x} + C_{r+1}e^{x a_{r+1}} + \dots + C_n e^{x a_n},$$

a single constant being substituted for the sum or product of several constants in forming the complementary function.

58. Again suppose

$$a_1 = \rho\,(\cos\theta + \sqrt{-1}\sin\theta) = \rho e^{\theta\sqrt{-1}}$$

to be an imaginary root of the auxiliary equation, then since A_1 is a function of a_1 it will be of the form

$$R\,(\cos\alpha + \sqrt{-1}\sin\alpha) = Re^{\alpha\sqrt{-1}},$$

and the term involving a_1 in the value of y will consequently be

$$Re^{x\rho\cos\theta}\,e^{(\alpha + x\rho\sin\theta)\sqrt{-1}}\int dx e^{-x\rho\cos\theta}\,e^{-x\rho\sin\theta\sqrt{-1}}\,X,$$

$$\text{or } Re^{x\rho\cos\theta}\,\{\cos(\alpha + x\rho\sin\theta) + \sqrt{-1}\sin(\alpha + x\rho\sin\theta)\}$$
$$\times \int dx e^{-x\rho\cos\theta}\,X\,\{\cos(x\rho\sin\theta) - \sqrt{-1}\sin(x\rho\sin\theta)\}.$$

Now the term in y involving the conjugate root to a_1, will result from this by changing the sign of $\sqrt{-1}$; and therefore the sum of the two terms introduced into the value of y by the pair of imaginary roots $\rho\,(\cos\theta \pm \sqrt{-1}\sin\theta)$, will equal twice the real part of the foregoing expression, that is,

$$2Re^{x\rho\cos\theta}\cos(\alpha + x\rho\sin\theta)\,\{\int dx e^{-x\rho\cos\theta}\cos(x\rho\sin\theta)\,X + C\}$$
$$+ 2Re^{x\rho\cos\theta}\sin(\alpha + x\rho\sin\theta)\,\{\int dx e^{-x\rho\cos\theta}\sin(x\rho\sin\theta)\,X + C'\}:$$

$$\text{where } 2Re^{x\rho\cos\theta}\,\{C\cos(\alpha + x\rho\sin\theta) + C'\sin(\alpha + x\rho\sin\theta)\},$$

are the terms introduced into the complementary function, and may be replaced by

$$e^{x\rho\cos\theta}\,\{c\cos(x\rho\sin\theta) + c'\sin(x\rho\sin\theta)\}.$$

59. Exactly in the same way, if the imaginary root

$$a_1 = \rho\,(\cos\theta + \sqrt{-1}\,\sin\theta) = \rho e^{\theta\sqrt{-1}}$$

occur r terms in the auxiliary equation, it will produce in the value of y, r terms of the form

$$A_m e^{a_1 x} \int^m dx^m e^{-a_1 x} X;$$

or since A_m is a function of a_1 and may be assumed

$$= R_m\,(\cos\alpha_m + \sqrt{-1}\,\sin\alpha_m) = R_m e^{\alpha_m\sqrt{-1}},$$

of the form $R_m e^{x\rho\cos\theta}\, e^{(\alpha_m + x\rho\sin\theta)\sqrt{-1}}\, (\int dx)^m\, e^{-x\rho\cos\theta}\, e^{-x\rho\sin\theta\sqrt{-1}}\, X$,

$$\text{or } R_m e^{x\rho\cos\theta}\{\cos(\alpha_m + x\rho\sin\theta) + \sqrt{-1}\,\sin(\alpha_m + x\rho\sin\theta)\}$$

$$\times\, (\int dx)^m\, e^{-x\rho\cos\theta}\{\cos(x\rho\sin\theta) - \sqrt{-1}\,\sin(x\rho\sin\theta)\}\,X.$$

But the root conjugate to a_1 will produce a term precisely the same as this except with $-\sqrt{-1}$ instead of $+\sqrt{-1}$; consequently the sum of these terms will produce twice the real part of the foregoing expression, that is,

$$2R_m e^{x\rho\cos\theta}\cos(\alpha_m + x\rho\sin\theta)\,(\int dx)^m e^{-x\rho\cos\theta}\cos(x\rho\sin\theta)\,X$$

$$+\,2R_m e^{x\rho\cos\theta}\sin(\alpha_m + x\rho\sin\theta)\,(\int dx)^m e^{-x\rho\cos\theta}\sin(x\rho\sin\theta)\,X;$$

and to get all the terms introduced into the value of y by the pair of imaginary roots $\rho\,(\cos\theta \pm \sqrt{-1}\,\sin\theta)$ that occur r times in the auxiliary equation, m in the above formula must receive all values from 1 to r. Also we see that the part of the complementary function introduced by these roots, by substituting a single constant for the sum or product of other arbitrary constants, will take the form

$$(c_0 + c_1 x + \ldots c_{r-1} x^{r-1})\, e^{x\rho\cos\theta}\cos(x\rho\sin\theta)$$

$$+\,(c_0' + c_1' x + \ldots + c'_{r-1} x^{r-1})\, e^{x\rho\cos\theta}\sin(x\rho\sin\theta).$$

60. We shall now give an example of each of the cases that have been examined. It may be observed that for equations capable of being reduced to either of the forms

$$f\left(\frac{d}{dx}\right) y = e^{mx},\quad f\left(\frac{d}{dx}\right) y = p_0 x^m + p_1 x^{m-1} + \ldots + p_m,$$

the process may be greatly simplified. For in the former case since $\frac{d}{dx} e^{mx} = me^{x}$, the symbol $\frac{d}{dx}$ is equivalent to the factor m; so that the solution of the equation $f\left(\frac{d}{dx}\right) y = e^{mx}$ is $y = \frac{e^{mx}}{f(m)}$. + complementary function. And in the latter case if $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$ can be expanded in a series of powers

$$A_0 + A_1 \frac{d}{dx} + A_2 \left(\frac{d}{dx}\right)^2 + \&c.,$$

then as every differential coefficient of X higher than the m^{th} is zero,

$$y = \left\{A_0 + A_1 \frac{d}{dx} + \ldots + A_m \left(\frac{d}{dx}\right)^m\right\} (p_0 x^m + p_1 x^{m-1} + \ldots + p_m).$$

Also if $X = \cos(mx + \alpha)$ or $\sin(mx + \alpha)$, since in either case $\frac{d^2X}{dx^2} = -m^2 X$, if the proposed equation contain only differential coefficients of an even order, that is, be of the form

$$f\left\{\left(\frac{d}{dx}\right)^2\right\} y = \cos(mx + \alpha),$$

then its solution is

$$y = \frac{\cos(mx + \alpha)}{f(-m^2)} + \text{complementary function.}$$

Ex. 1. $\frac{d^3y}{dx^3} - 6n \frac{d^2y}{dx^2} + nn^2 \frac{dy}{dx} - 6n^3 y = a^x,$

the roots of the auxiliary equations are n, $2n$, $3n$; and $a^x = e^{x \log a}$;

$$\therefore\ y = \frac{a^x}{(\log a - n)(\log a - 2n)(\log a - 3n)} + c_0 e^{nx} + c_1 e^{2nx} + c_2 e^{3nx}.$$

Ex. 2. $\frac{d^4y}{dx^4} - 10 \frac{d^3y}{dx^3} + 62 \frac{d^2y}{dx^2} - 210 \frac{dy}{dx} + 261 y = e^x,$

the roots of the auxiliary equation are $2 \pm 5\sqrt{-1}$, 3, 3; and $f(1) = 104$,

$$\therefore\ y = \frac{e^x}{104} + (a + a'x) e^{3x} + (b \cos 5x + b' \sin 5x) e^{2x}.$$

Ex. 3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = ax^3$,

$$\therefore\ y = \left(1 + \frac{d}{dx}\right)^{-2} ax^3 = \left\{1 - 2\frac{d}{dx} + 3\left(\frac{d}{dx}\right)^2 - 4\left(\frac{d}{dx}\right)^3\right\} ax^3 + \&c.$$

$$= a\,(x^3 - 6x^2 + 18x - 24) + (c + c'x)e^{-x}.$$

Ex. 4. $\dfrac{d^4y}{dx^4} + 5\left(\dfrac{dy}{dx}\right)^2 + 6y = \cos\,(mx + \alpha)$.

$$\text{or}\ \left\{\left(\frac{d}{dx}\right)^2 + 2\right\}\left\{\left(\frac{d}{dx}\right)^2 + 3\right\} y = \cos\,(mx + \alpha).$$

$$\therefore\ y = \frac{\cos\,(mx + \alpha)}{m^4 - 5m^2 + 6} + C\cos\,(x\sqrt{2} + \beta) + C'\cos\,(x\sqrt{3} + \gamma).$$

Ex. 5. $\dfrac{d^ny}{dx^n} - a^ny = X$.

Let $b = a\,(\cos\phi + \sqrt{-1}\sin\phi) = ae^{\phi\sqrt{-1}}$ be a root of the auxiliary equation $z^n - a^n = 0$, where $\phi = \dfrac{2r\pi}{n}$, r being any integer, then if $B\,(z - b)^{-1}$ be the term corresponding to this root in the resolution of $(z^n - a^n)^{-1}$ into partial fractions,

$$B = \frac{1}{nb^{n-1}} = \frac{e^{-(n-1)\phi\sqrt{-1}}}{na^{n-1}}.$$

Hence the corresponding term in the value of $na^{n-1}y$ is

$$e^{-(n-1)\phi\sqrt{-1}}\left(\frac{d}{dx} - b\right)^{-1} X = e^{-(n-1)\phi\sqrt{-1}} e^{bx}\int dx e^{-bx} X$$

$$= e^{ax\cos\phi} e^{(ax\sin\phi - n\phi + \phi)\sqrt{-1}}\int dx\, e^{-ax\cos\phi} e^{-ax\sin\phi\sqrt{-1}} X$$

$$= e^{ax\cos\phi}\{\cos\,(ax\sin\phi - n\phi + \phi) + \sqrt{-1}\sin\,(ax\sin\phi - n\phi + \phi)\}$$

$$\times\int dx\, e^{-ax\cos\phi}\{\cos\,(ax\sin\phi) - \sqrt{-1}\sin\,(ax\sin\phi)\}X;$$

and as the root conjugate to b will produce a term exactly the same as this, only with $-\sqrt{-1}$ instead of $+\sqrt{-1}$, doubling the real part of this expression, we get

$$\frac{2e^{ax\cos\phi}}{na^{n-1}}\cos(ax\sin\phi - n\phi + \phi)\int dx\cos(ax\sin\phi)\,Xe^{-ax\cos\phi}$$

$$+\frac{2e^{ax\cos\phi}}{na^{n-1}}\sin(ax\sin\phi - n\phi + \phi)\int dx\sin(ax\sin\phi)\,Xe^{-ax\cos\phi}$$

for the general term in the value of y, where $\phi = \frac{1}{n}2r\pi$, and to get all the terms, r must be taken from 0 to $\frac{1}{2}(n-1)$ or $\frac{1}{2}n$, according as n is odd or even; only the terms that result from this by making $r=0$ or $r=\frac{1}{2}n$, must be divided by 2.

Ex. 6. $\frac{d^ny}{dx^n}+a^ny=X$. The process, and the expression for the general term in the value of y, are exactly the same as in the preceding example, except that $\phi=\frac{1}{n}(2r+1)\pi$, where r is any integer; and to get all the terms we must take r from 0 to $\frac{1}{2}(n-1)$ or $\frac{1}{2}n-1$ according as n is odd or even; only the term that results from taking $r=\frac{1}{2}(n-1)$ must be divided by 2.

Ex. 7. $\frac{d^{2n}y}{dx^{2n}}-2\cos\theta\, a^n\frac{d^ny}{dx^n}+a^{2n}y=X$.

Let $b=a(\cos\phi+\sqrt{-1}\sin\phi)=ae^{\phi\sqrt{-1}}$ be a root of the auxiliary equation $z^{2n}-2\cos\theta a^nz^n+a^{2n}=0$, where $\phi=\frac{1}{n}(2r\pi+\theta)$ r being any integer. Then if $B(z-b)^{-1}$ be the term corresponding to this root in the resolution of $(z^{2n}-2\cos\theta a^nz^n+a^{2n})^{-1}$, into partial fractions, since $b^n=a^n(\cos\theta+\sqrt{-1}\sin\theta)$,

$$B=\frac{1}{2n(b^n-a^n\cos\theta)\,b^{n-1}}=\frac{1}{2n\sqrt{-1}\sin\theta a^nb^{n-1}}=\frac{e^{-(n-1)\phi\sqrt{-1}}}{2n\sqrt{-1}\sin\theta a^{2n-1}}.$$

Hence the corresponding term in the value of $n\sin\theta a^{2n-1}y$ is

$$\frac{1}{2\sqrt{-1}}e^{-(n-1)\phi\sqrt{-1}}\left(\frac{d}{dx}-b\right)^{-1}X=\frac{1}{2\sqrt{-1}}e^{-(n-1)\phi\sqrt{-1}}e^{bx}\int dxe^{-bx}X,$$

$$\text{or }\frac{1}{2\sqrt{-1}}e^{ax\cos\phi}e^{(ax\sin\phi-n\phi+\phi)\sqrt{-1}}\int dxe^{-ax\cos\phi}e^{-ax\sin\phi\sqrt{-1}}X,$$

$$\text{or } \frac{1}{2\sqrt{-1}} e^{ax\cos\phi} \{\cos(ax\sin\phi - n\phi + \phi) + \sqrt{-1}\sin(ax\sin\phi - n\phi + \phi)\}$$

$$\times \int dx e^{-ax\cos\phi} \{\cos(ax\sin\phi) - \sqrt{-1}\sin(ax\sin\phi)\} X.$$

Hence taking double the real part of this expression we get for the general term of the value of $n \sin\theta a^{2n-1} y$

$$e^{ax\cos\phi} \sin(ax\sin\phi - n\phi + \phi) \int dx e^{-ax\cos\phi} \cos(ax\sin\phi) X$$

$$-e^{ax\cos\phi} \cos(ax\sin\phi - n\phi + \phi) \int dx e^{-ax\cos\phi} \sin(ax\sin\phi) X,$$

when $\phi = \frac{1}{n}(2r\pi + \theta)$, and to get all the terms, r must be taken from 0 to $n-1$.

61. The solution of $f\left(\frac{d}{dx}\right) y = e^{mx}$ in the form

$$y = \frac{e^{mx}}{f(m)} + c_1 e^{a_1 x} + \&c.,$$

fails, when $m = a_1$ a root of the auxiliary equation. Suppose that a_1, the root to which m becomes equal, occurs twice in the auxiliary equation, so that $f(a_1) = f'(a_1) = 0$; then by altering the constants the value of y may be written

$$y = \frac{1}{f(m)} (e^{mx} - e^{a_1 x} - hxe^{a_1 x}) + (c + c'x) e^{a_1 x} + \&c.$$

Now let $m = a_1 + h$ where h is a small quantity, then

$$y = e^{a_1 x} . \frac{\frac{1}{2}h^2x^2 + \frac{1}{6}h^3x^3 + \&c.}{\frac{1}{2}h^2 f''(a_1) + \&c.} + (c + c'x) e^{a_1 x} + \&c.,$$

or, making $h = 0$,

$$y = \frac{x^2 e^{a_1 x}}{f''(a_1)} + (c + c'x) e^{a_1 x} + \&c.$$

And if the root a_1 to which m becomes equal, occur r times,

$$y = \frac{x^r e^{a_1 x}}{f^{(r)}(a_1)} + (c_0 + c_1 x + \ldots + c_{r-1} x^{r-1}) e^{a_1 x} + \&c.$$

These results are easily obtained by differentiating the

numerator and denominator of the vanishing fraction $\dfrac{e^{mx} - e^{a_1 x}}{f(m)}$ in the usual way.

The principle employed here and in all similar cases is to suppose the arbitrary constant to have such a value as to make the expression which becomes infinite, assume the indeterminate form $\dfrac{0}{0}$; then its real value can be assigned by the ordinary rules. In the instances which follow, a negative value is assumed for the arbitrary constant, which becomes infinite; so that the difference between the two infinite quantities may be finite.

Ex. 1. $\dfrac{d^4y}{dx^4} - a^4y = e^{ax}$. First suppose the second member to be e^{mx}, then

$$y = \frac{e^{mx} - e^{ax}}{m^4 - a^4} + c_1e^{ax} + c_2e^{-ax} + \beta \cos(ax + \alpha);$$

hence making $m = a$,

$$y = \frac{xe^{ax}}{4a^3} + c_1e^{ax} + c_2e^{-ax} + \beta \cos(ax + \alpha).$$

Ex. 2. $\dfrac{d^4y}{dx^4} - a^4y = \cos ax$. Suppose the second member to be $\cos mx$, then

$$y = \frac{\cos mx - \cos ax}{m^4 - a^4} + c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax;$$

hence making $m = a$,

$$y = -\frac{x \sin ax}{4a^3} + \text{complementary function.}$$

Ex. 3. $\dfrac{d^2y}{dx^2} + n^2y = \cos nx$,

$$y = \frac{x \sin nx}{2n} + c \cos nx + c' \sin nx.$$

62. To deduce the solution of the linear equation of the n^{th} order with variable coefficients

$$\frac{d^ny}{dx^n} + p_1 \frac{d^{n-1}y}{dx^{n-1}} + p_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + p_ny = X,$$

from the solution of the same equation when the term independent of y is supposed to become zero.

Let $u_1, u_2, \ldots u_n$ be the n particular values of u which satisfy the equation

$$\frac{d^n u}{dx^n} + p_1 \frac{d^{n-1}u}{dx^{n-1}} + p_2 \frac{d^{n-2}u}{dx^{n-2}} + \ldots + p_n u = 0 \ldots\ldots\ldots (1),$$

with which the proposed coincides if we suppose its second member to become zero; then, as already proved,

$$u = c_1 u_1 + c_2 u_2 + \ldots + c_n u_n.$$

If we now divide both sides by u_1 and differentiate, we shall eliminate c_1; next dividing both sides by the coefficient of c_2 in the new result which suppose v_1, and differentiating, we shall eliminate c_2; again dividing by v_2 the coefficient which now affects c_3 and differentiating, we shall eliminate c_3; and proceeding in this manner till all the constants are eliminated, our final result will be of the form (where each symbol of differentiation affects the whole of the expression after it)

$$\frac{d}{dx}\frac{1}{v_{n-1}}\frac{d}{dx}\frac{1}{v_{n-2}} \cdots \frac{d}{dx}\frac{1}{v_1}\frac{d}{dx}\frac{u}{u_1} = 0,$$

in which the coefficient of $\frac{d^n u}{dx^n}$ is evidently $\frac{1}{v_{n-1}v_{n-2}\ldots v_1 u_1}$; therefore dividing by this, so as to make the coefficient of $\frac{d^n u}{dx^n}$ unity, we get

$$v_{n-1}v_{n-2}\ldots v_1 u_1 \frac{d}{dx}\frac{1}{v_{n-1}}\frac{d}{dx}\frac{1}{v_{n-2}} \cdots \frac{d}{dx}\frac{1}{v_1}\frac{d}{dx}\frac{u}{u_1},$$

which must be equivalent to the first member of equation (1). Therefore the same expression, only with y instead of u, must be equivalent to the first member of the proposed equation, and therefore equal to X. Hence equating these equals, and reversing the operations, we find the integral of the proposed equation,

$$y = u_1 \int dx\, v_1 \int dx\, v_2 \ldots \int dx\, v_{n-1} \int dx \frac{X}{u_1 v_1 v_2 \ldots v_{n-1}};$$

(each symbol of integration affecting the whole of the expression which follows it), which is a general formula for deducing the solution of a complete linear equation of the n^{th} order, from the

solution of the same equation when the term independent of y is supposed to become zero.

63. The application of this method is seen in the following examples.

Ex. 1. $\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+4y=ax^2$.

If we suppose u to denote the value of y in this equation when $a=0$, since the roots of the auxiliary equation are -2, -2,

$$u=e^{-2x}(c_1+c_2x)\,; \quad \therefore\ \frac{d^2}{dx^2}(e^{2x}u)=0,$$

in which expression the coefficient of $\dfrac{d^2u}{dx^2}$ is e^{2x}, therefore dividing by this coefficient, and changing u into y, we get

$$e^{-2x}\frac{d^2}{dx^2}(e^{2x}y)=ax^2\,;$$

$$\therefore\ y=e^{-2x}\textstyle\int^2 dx^2\,(ax^2e^{2x})=a\left(\dfrac{1}{4}x^2-\dfrac{1}{2}x+\dfrac{3}{8}\right)+e^{-2x}(c_1+c_2x).$$

Ex. 2. $\dfrac{d^2y}{dx^2}-2m\dfrac{dy}{dx}+(m^2+n^2)\,y=X$.

If we suppose u to be the value of y in this equation when $X=0$, we have

$$u=C_1e^{mx}\cos nx+C_2e^{mx}\sin nx\,;$$

$\therefore$ dividing by $e^{mx}\cos nx$, and differentiating, we find

$$\frac{d}{dx}e^{-mx}\sec nx\,.\,u=C_2n\sec^2 nx\,;$$

$$\therefore\ \frac{d}{dx}\cos^2 nx\,\frac{d}{dx}e^{-mx}\sec nx\,.\,u=0,$$

in which expression the coefficient of $\dfrac{d^2u}{dx^2}$ is $e^{-mx}\cos nx$, therefore dividing by this coefficient and changing u into y, we get

$$e^{mx}\sec nx\,\frac{d}{dx}\cos^2 nx\,\frac{d}{dx}e^{-mx}\sec nx\,.\,y=X\,;$$

$$\therefore\ y=e^{mx}\cos nx\textstyle\int dx\sec^2 nx\int dx e^{-mx}\cos nxX,$$

or, if we suppose the constants to be added after each integration,

$$y=e^{mx}(C\cos nx+C'\sin nx)+e^{mx}\cos nx\textstyle\int dx\sec^2 nx\int dx e^{-mx}\cos nx\,X.$$

Hence the solution of $\dfrac{d^2y}{dx^2}+n^2y=X$ is

$$y=C\cos nx+C'\sin nx+\cos nx\int dx\sec^2 nx\int dx\cos nxX.$$

Also if we take $X=A\cos(mx+\alpha)+B\cos(nx+\beta)$,

then $\int dx\cos nxX$

$$=\frac{A}{m^2-n^2}\{m\sin(mx+\alpha)\cos nx-n\cos(mx+\alpha)\sin nx\}$$

$$+\frac{B}{2}\left\{\frac{1}{2n}\sin(2nx+\beta)+x\cos\beta\right\};$$

$$\therefore\ \int dx\sec^2 nx\int dx\cos nxX=-\frac{A}{m^2-n^2}\,\frac{\cos(mx+\alpha)}{\cos nx}$$

$$+\frac{B}{2}\left\{\frac{x\sin(nx+\beta)}{n\cos nx}-\frac{\sin\beta}{2n^2}\tan nx\right\};$$

$$\therefore\ y=C\cos nx+C'\sin nx+\frac{A}{n^2-m^2}\cos(mx+\alpha)$$

$$+\frac{B}{2n}x\sin(nx+\beta).$$

Ex. 3. $\dfrac{d^2y}{dx^2}+2n\dfrac{dy}{dx}-\dfrac{2y}{x^2}=\dfrac{a}{x}\left(nx+\dfrac{1}{nx}-1\right)$;

here $u=c_1\left(1-\dfrac{1}{nx}\right)+c_2\left(1+\dfrac{1}{nx}\right)e^{-2nx}$, (Ex. 2. Art. 64.)

and $y=(\tfrac{1}{2}ax+C')\left(1-\dfrac{1}{nx}\right)+C\left(1+\dfrac{1}{nx}\right)e^{-2nx}$.

64. As this method cannot be applied till we have obtained a solution of the proposed equation deprived of the term independent of y, the following propositions are sometimes of use for equations of the second order.

If $y=u$ be a particular integral of the equation

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0,$$

the other particular integral will be $u\int dx\,(u^{-2}e^{-\int dxP})$. For if we denote the two values of y by u and v, we have

$$\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu=0,\quad \frac{d^2v}{dx^2}+P\frac{dv}{dx}+Qv=0,$$

$$\therefore\ u\frac{d^2v}{dx^2} - v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) = 0,$$

$$\therefore\ u\frac{dv}{dx} - v\frac{du}{dx} = Ce^{-\int dxP}, \text{ or } \frac{d}{dx}\left(\frac{v}{u}\right) = Cu^{-2}e^{-\int dxP};$$

$$\therefore\ v = Cu\int dx\,(u^{-2}e^{-\int dxP}).$$

Ex. 1. $\dfrac{d^2y}{dx^2} + mP\dfrac{dy}{dx} - m^2(P+1)\,y = 0.$

It is easily seen that $y = u = e^{mx}$ is a solution;

$$\therefore\ y = e^{mx}\,(C' + C\int dx\,e^{-2mx-\int dxP}).$$

Suppose $m = 1,\quad P = -\dfrac{2}{x+3},\quad -\int dxP = \log(x+3)^2;$

$$\therefore\ y = e^x\{C' + C\int dx\,e^{-2x}(x+3)^2\} = C'e^x + Ce^{-x}(x^2 + 7x + 12\tfrac{1}{2})$$

is the solution of

$$(x+3)\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - (x+1)\,y = 0.$$

Ex. 2. $\dfrac{d^2y}{dx^2} + 2n\dfrac{dy}{dx} - \dfrac{2y}{x^2} = 0$, of which $y = u = 1 - \dfrac{1}{nx}$ is a solution, and thence we get $y = C'\left(1 - \dfrac{1}{nx}\right) + C\left(1 + \dfrac{1}{nx}\right)e^{-2nx}.$

Ex. 3. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\left(n + \dfrac{m}{1+x^m}\right)\dfrac{dy}{dx} - \dfrac{m+1}{4x^2}\left(m + 2n - 1 + \dfrac{2m}{1+x^m}\right)y = 0;$

$y = x^{\frac{m+1}{2}}$, from which the complete integral may be obtained.

Ex. 4. $\dfrac{d^2y}{dx^2} + \dfrac{5+2x^3}{x+x^4}\,\dfrac{dy}{dx} - \dfrac{12+6x^3}{x^2+x^5}\,y = 0;$

of which $y = u = x^2$ is a solution.

$$P = -\frac{-5x^{-6} - 2x^{-3}}{x^{-5} + x^{-2}},\quad \therefore\ \int dxP = -\log(x^{-5} + x^{-2})$$

$$\therefore\ \frac{1}{u^2}e^{-\int dxP} = x^{-9} + x^{-6},\quad \therefore\ u\int dx\frac{e^{-\int dxP}}{u^2} = -\frac{x^{-6}}{8} - \frac{x^{-3}}{5},$$

$$y = Cx^2 + C'\left(\frac{1}{x^6} + \frac{8}{5}\,\frac{1}{x^3}\right).$$

65. It is easily seen, by actual substitution, that if $y = u$ satisfy the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + n^2y = 0,$$

then the equation

$$\frac{d^2y}{dx^2} - P\frac{dy}{dx} + n^2y = 0$$

will be satisfied by the value

$$y = \frac{du}{dx}e^{\int dxP}.$$

Thus for the equation $\dfrac{d^2y}{dx^2} + \dfrac{4}{x}\dfrac{dy}{dx} + n^2y = 0$

$$y = u = Cx^{-3}\{\cos(nx + \alpha) + nx\sin(nx + \alpha)\};$$

$$\therefore\quad \frac{du}{dx} = Cx^{-4}\{(n^2x^2 - 3)\cos(nx + \alpha) - 3nx\sin(nx + \alpha)\},$$

$$\text{and } e^{\int dxP} = x^4,$$

$$\therefore\quad y = C\{(n^2x^2 - 3)\cos(nx + \alpha) - 3nx\sin(nx + \alpha)\}$$

is the complete solution of $\dfrac{d^2y}{dx^2} - \dfrac{4}{x}\dfrac{dy}{dx} + n^2y = 0.$

66. If we know a particular integral of a linear equation of any order that has no term independent of y, we may reduce it to another equation of the same kind, of the order immediately inferior.

Let $y = u$ be a particular integral of $f\left(\dfrac{d}{dx}\right)y = 0,$

$$\text{or } \frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + p_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_ny = 0, \ldots\ldots\ldots (1),$$

and assume $y = u\int dxz$, z being a new variable, a function of x; then, separating the symbols of operation from those of quantity, we have

$$\frac{d^ny}{dx^n} = \left(\frac{d}{dx} + \frac{d'}{dx}\right)^n u\int dxz,$$

$\frac{d}{dx}$ being understood to affect u only, and $\frac{d'}{dx}$ to affect $\int dxz$ only.

Hence, substituting for the differential coefficients in the proposed, by this formula, and again separating the symbols of operation from those of quantity, we get successively

$$\left(\frac{d}{dx}+\frac{d'}{dx}\right)^n u\int dxz + p_1\left(\frac{d}{dx}+\frac{d'}{dx}\right)^{n-1} u\int dxz + \&c. + p_n u\int dxz$$

$$= f\left(\frac{d}{dx}+\frac{d'}{dx}\right) u\int dxz$$

$$= \int dxz \,.\, f\left(\frac{d}{dx}\right) u + zf'\left(\frac{d}{dx}\right) u + \frac{1}{1\,.\,2}\,\frac{dz}{dx}\,.\,f''\left(\frac{d}{dx}\right) u + \ldots$$

$$+ \frac{1}{\underline{|n-1}}\,\frac{d^{n-2}z}{dx^{n-2}} f^{(n-1)}\left(\frac{d}{dx}\right) u + \frac{d^{n-1}z}{dx^{n-1}}\,.\,u.$$

But since u is a particular value of y, $f\left(\frac{d}{dx}\right) u = 0$; hence reversing the order of the terms and observing that $f^{(r)}\left(\frac{d}{dx}\right)$ denotes the same function of the symbol $\frac{d}{dx}$, that $\frac{d^r f(v)}{dv^r}$ does of v, we get for the depressed equation

$$u\frac{d^{n-1}z}{dx^{n-1}} + \left(n\frac{du}{dx} + p_1 u\right)\frac{d^{n-2}z}{dx^{n-2}}$$

$$+ \left\{\frac{n(n-1)}{1\,.\,2}\,\frac{d^2u}{dx^2} + (n-1)\,p_1\frac{du}{dx} + p_2 u\right\}\frac{d^{n-3}z}{dx^{n-3}} + \ldots + zf'\left(\frac{d}{dx}\right) u = 0 \ldots (2)$$

Similarly, if we know another particular solution u_1 of equation (1), then $\frac{d}{dx}\left(\frac{u_1}{u}\right)$ will be a value of z in equation (2), and we may depress this equation to another of the same form of the $(n-2)^{\text{th}}$ order; and if we know r particular solntions of equation (1), we may in this way depress it to an equation of the same form of the $(n-r)^{\text{th}}$ order.

Method of Parameters.

67. There is also another mode of integrating linear equations, which deserves to be mentioned, called the method of Parameters, which we shall now explain. It may be stated thus. The complete integral of the linear equation of the n^{th} order will be of the form

$$y = v_1 u_1 + v_2 u_2 + \dots + v_n u_n,$$

where $u_1, u_2, \dots u_n$ are the n particular integrals of the equation when the term independent of y becomes zero; and $v_1, v_2, \dots v_n$ are functions of x, determined by equations of the form

$$v_1 = \int dx f_1(x) + C_1.$$

Let the proposed equation be

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + p_n y = X,$$

and suppose $y = v_1 u_1 + v_2 u_2 + \dots + v_n u_n = \Sigma\,(vu)$,

and as we have made only one assumption respecting the n independent quantities $v_1, v_2 \dots v_n$, we may make $n-1$ more;

$$\text{now } \frac{dy}{dx} = \Sigma\left(v\frac{du}{dx}\right) + \Sigma\left(u\frac{dv}{dx}\right); \quad \therefore \frac{dy}{dx} = \Sigma\left(v\frac{du}{dx}\right),$$

putting for the first of our additional assumptions $\Sigma\left(u\dfrac{dv}{dx}\right) = 0$, (1);

similarly $\dfrac{d^2y}{dx^2} = \Sigma\left(v\dfrac{d^2u}{dx^2}\right)$, putting $\Sigma\left(\dfrac{du}{dx}\dfrac{dv}{dx}\right) = 0$, (2), &c. = &c.

and $\dfrac{d^n y}{dx^n} = \Sigma\left(v\dfrac{d^n u}{dx^n}\right) + X$, putting $\Sigma\left(\dfrac{d^{n-1}u}{dx^{n-1}}\dfrac{dv}{dx}\right) = X$, ($n$).

Substitute these values for $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dots \dfrac{d^n y}{dx^n}$, in the proposed equation, and the first member becomes

$$v_1\left(\frac{d^n u_1}{dx^n} + p_1\frac{d^{n-1}u_1}{dx^{n-1}} + \dots + p_n u_1\right) + v_2\left(\frac{d^n u_2}{dx^n} + p_1\frac{d^{n-1}u_2}{dx^{n-1}} + \dots + p_n u_2\right)$$

$$+ \&\text{c.} + v_n\left(\frac{d^n u_n}{dx^n} + p_1\frac{d^{n-1}u_n}{dx^{n-1}} + \dots + p_n u_n\right) + X,$$

which manifestly reduces itself to X, since each of the quantities within brackets is equal to zero; hence if the parameters $v_1, v_2, \dots v_n$ be determined subject to the above n equations of condition, the assumed value of y will satisfy the equation; and since there are n equations in which the n quantities

$$\frac{dv_1}{dx}, \frac{dv_2}{dx}, \dots \frac{dv_n}{dx}$$

enter to the first degree, each of the parameters v_1, v_2, &c. will be determined by an equation of the form $\frac{dv_1}{dx} = f_1(x)$, and so n arbitrary constants will be introduced.

Ex. $\dfrac{d^2y}{dx^2} + n^2y = \sec nx.$

The solution of $\dfrac{d^2y}{dx^2} + n^2y = 0$ is $y = a_1 \cos nx + a_2 \sin nx$.

Let $\therefore\ y = v_1 \cos nx + v_2 \sin nx$;

$$\therefore \frac{dy}{dx} = -v_1 n \sin nx + v_2 n \cos nx,$$

$$\text{making } \frac{dv_1}{dx} \cos nx + \frac{dv_2}{dx} \sin nx = 0.$$

$$\frac{d^2y}{dx^2} = -v_1 n^2 \cos nx - v_2 n^2 \sin nx + \sec nx,$$

$$\text{making } -\frac{dv_1}{dx} n \sin nx + \frac{dv_2}{dx} n \cos nx = \sec nx;$$

$$\therefore \frac{dv_1}{dx} = -\frac{1}{n}\frac{\sin nx}{\cos nx}, \quad \text{or } v_1 = \frac{1}{n^2}\log \cos nx + C_1.$$

$$\frac{dv_2}{dx} = \frac{1}{n}, \quad \text{or } v_2 = \frac{x}{n} + C_2;$$

$$\therefore y = \left(\frac{1}{n^2}\log \cos nx + C_1\right)\cos nx + \left(\frac{x}{n} + C_2\right)\sin nx.$$

Change of the Independent Variable.

68. Besides linear equations with constant coefficients, very few equations of the second and superior orders admit of being integrated; and those only by particular methods. Sometimes

the integration of an equation may be facilitated by changing the independent variable; and this is the method which has been attended with the greatest success. Thus in the example

$$\frac{d^2y}{dx^2} + a\left(\frac{dy}{dx}\right)^2 + bx\left(\frac{dy}{dx}\right)^3 = 0,$$

if we make y the independent variable, and consequently for

$$\frac{dy}{dx}, \ \frac{d^2y}{dx^2}, \text{ write } \frac{1}{\frac{dx}{dy}}, \ -\frac{d^2x}{dy^2} \div \left(\frac{dy}{dx}\right)^3,$$

the equation is transformed into the integrable shape

$$\frac{d^2x}{dy^2} - a\frac{dx}{dy} - bx = 0.$$

69. In the above example there is no difficulty in fixing upon the new independent variable. In other cases we must consider x and y as functions of a third variable t, and substitute the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, &c. corresponding to that supposition. When the equation is thus generalized, we must assume for x or y some known function of t, adapted to the particular form of the equation, so that there may arise for determining the function of t which expresses y or x, a differential equation simpler than the proposed one; between the integral of which and the assumed function, if we eliminate t, we obtain the required relation between x and y.

Ex. 1. $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0.$

The generalized equation is

$$(1 - x^2)\frac{x'y'' - y'x''}{x'^3} - x\frac{y'}{x'} + n^2y = 0,$$

using x', x'', &c. to denote $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, &c.

Let $x = \cos t$, $\therefore x' = -\sin t$, $x'' = -\cos t$, therefore by substitution $\frac{d^2y}{dt^2} + n^2y = 0$, which gives

$$y = A\cos nt + B\sin nt;$$

$$\therefore\; y = A\cos(n\cos^{-1}x) + B\sin(n\cos^{-1}x).$$

Ex. 2. $x^4\dfrac{d^2y}{dx^2} + cy = 0.$

The generalized equation is

$$x^4 . \frac{x'y'' - y'x''}{x'^3} + cy = 0\,;$$

$$\text{let } x' = -x^2, \text{ or } x = \frac{1}{t}, \quad \therefore\; x'' = 2x^3,$$

hence by substitution we get

$$y'' + \frac{2}{t}y' + cy = 0, \quad \text{or } \frac{d^2(ty)}{dt^2} + cty = 0\,;$$

$$\therefore\; ty = \beta\cos(t\sqrt{c} + \alpha), \quad \text{or } y = \beta x\cos\left(\frac{\sqrt{c}}{x} + \alpha\right).$$

If $c = -n^2$, the solution is $y = x\left(c_1e^{\frac{n}{x}} + c_2e^{-\frac{n}{x}}\right)$.

Ex. 3. $u^2\dfrac{d^2y}{dx^2} + \left(\dfrac{du}{dx} + a\right)u\dfrac{dy}{dx} + by = X,$

where u denotes a given function of x. This when generalized becomes

$$u^2 . \frac{x'y'' - y'x''}{x'^3} + \left(\frac{du}{dx} + a\right)u\frac{y'}{x'} + by = X\,;$$

$$\text{let } x' = u, \text{ or } t = \int dx\frac{1}{u}, \quad \therefore\; \frac{du}{dx} = \frac{x''}{x'},$$

hence by substitution we get the integrable form

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = X.$$

Ex. 4. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = \dfrac{ax^2}{x^2 - 1}.$

Here the substitution $x = e^t$ gives

$$\frac{d^2y}{dt^2} - y = a + \frac{a}{e^{2t} - 1}\,;$$

$$\therefore\ y = c_1 e^t + c_2 e^{-t} + \tfrac{1}{2} a + \tfrac{1}{4} a\,(e^{-t} - e^t) \log \left(\frac{e^t - 1}{e^t + 1}\right)$$

$$= c_1 x + \frac{c_2}{x} + \tfrac{1}{2} a + \tfrac{1}{4} a \left(\frac{1}{x} - x\right) \log \left(\frac{x-1}{x+1}\right).$$

Ex. 5. $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = ax^n$,

$$y = c_1 x + \frac{c_2}{x} + \frac{ax^n}{n^2 - 1}.$$

Ex. 6. $\dfrac{d^2y}{dx^2} + \dfrac{A}{a+bx}\dfrac{dy}{dx} + \dfrac{B}{(a+bx)^2} y = 0.$

The generalized equation is

$$\frac{x'y'' - y'x''}{x'^3} + \frac{A}{a+bx}\frac{y'}{x'} + \frac{B}{(a+bx)^2} y = 0.$$

Let $x' = a + bx$, or $e^{bt} = a + bx$; $\therefore\ x'' = bx'$;

hence, substituting,

$$\frac{d^2y}{dt^2} + (A - b)\frac{dy}{dt} + By = 0;$$

the solution of which is

$$y = c_1 e^{mt} + c_2 e^{m't} = c_1 (a+bx)^{\frac{m}{b}} + c_2 (a+bx)^{\frac{m'}{b}},$$

m and m' being the real roots of $m^2 + (A - b)\,m + B = 0$.

If the roots be impossible, and of the form $m \pm n\sqrt{-1}$, the solution is

$$y = \beta e^{mt} \cos(nt + \alpha) = \beta\,(a+bx)^{\frac{m}{b}} \cos\{\log(a+bx)^{\frac{n}{b}} + \alpha\}.$$

The same substitution of course succeeds for the equation

$$(a+bx)^n \frac{d^ny}{dx^n} + A\,(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \text{\&c.}$$

$$+ K(a+bx)\frac{dy}{dx} + Ly = f(x),$$

which, by putting $a+bx=z$, may first of all be reduced to the form

$$z^n\frac{d^ny}{dz^n}+a_1z^{n-1}\frac{d^{n-1}y}{dz^{n-1}}+\&c.+a_{n-1}z\frac{dy}{dz}+a_ny=f_1(z);$$

and then to a linear equation with constant coefficients, by the formula

$$z^n\frac{d^ny}{dz^n}=\frac{d^ny}{dt^n}-p_1\frac{d^{n-1}y}{dt^{n-1}}+\&c.\pm p_{n-1}y,$$

where $z=e^t$, and p_1, p_2, &c. are such that the roots of

$$k^n-p_1k^{n-1}+p_2k^{n-2}-\&c.\ldots\pm p_{n-1}k=0$$

are 0, 1, 2, 3 ... $(n-1)$; so that the preceding formula may be written

$$z^n\frac{d^ny}{dz^n}=\left\{\left(\frac{d}{dt}\right)^n-p_1\left(\frac{d}{dt}\right)^{n-1}+\ldots\pm p_n\right\}y$$

$$=\left\{\frac{d}{dt}\left(\frac{d}{dt}-1\right)\left(\frac{d}{dt}-2\right)\ldots\left(\frac{d}{dt}-n+1\right)\right\}y.$$

70. The result just noticed

$$x^n\frac{d^ny}{dx^n}=\frac{d}{dt}\left(\frac{d}{dt}-1\right)\ldots\left(\frac{d}{dt}-n+1\right)y,$$

where $x=e^t$, may be written

$$x^n\frac{d^ny}{dx^n}=P_n\left(\frac{d}{dt}\right)y,$$

using the symbol P_n to denote the product of the n factors formed by subtracting from $\frac{d}{dt}$ the n numbers 0, 1, 2, ... $n-1$. Hence multiplying by $x^m=e^{mt}$, we get

$$x^{m+n}\frac{d^ny}{dx^n}=e^{mt}P_n\left(\frac{d}{dt}+m-m\right)y.$$

But by Art. 53 we have

$$f\left(\frac{d}{dt}\right)e^{mt}y=e^{mt}f\left(\frac{d}{dt}+m\right)y,$$

which shews that in the expression $f\left(\frac{d}{dt}\right) e^{mt}y$ we may remove the power of e^t from the operation of the symbol $f\left(\frac{d}{dt}\right)$, and prefix it as a factor of the whole expression, provided we add the index of the power to $\frac{d}{dt}$; and conversely in the expression $e^{mt} f\left(\frac{d}{dt} + m\right) y$ we may transfer the power of e^t to be a factor of y, and so bring it under the symbol of operation, by subtracting the index of the power from $\frac{d}{dt} + m$. Hence

$$x^{m+n} \frac{d^n y}{dx^n} = P_n \left(\frac{d}{dt} - m\right) e^{mt} y.$$

71. Hence an expression such as

$$(a + bx + cx^2 + \ldots + lx^r)\, x^n \frac{d^n y}{dx^n}$$

is transformed by the substitution $x = e^t$ into

$$aP_n(d)\, y + bP_n(d-1)\, e^t y + cP_n(d-2)\, e^{2t} y + \ldots + lP_n(d-r)\, e^{rt} y,$$

where for convenience d is written for $\frac{d}{dt}$, the differentials of t being omitted; and consequently an equation such as

$$(a + bx + cx^2 + \&c.)\, x^n \frac{d^n y}{dx^n} + (a' + b'x + c'x^2 + \&c.)\, x^{n-1} \frac{d^{n-1} y}{dx^{n-1}}$$

$$+ (a'' + b''x + \&c.)\, x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + (a^{(n)} + b^{(n)}x + \&c.)\, y = X$$

is transformed into

$$aP_n(d)\, y + a'P_{n-1}(d)\, y + a''P_{n-2}(d)\, y + \ldots + a^{(n)} y$$

$$+ bP_n(d-1)\, e^t y + b'P_{n-1}(d-1)\, e^t y + b''P_{n-2}(d-1)\, e^t y + \ldots + b^{(n)} e^t y$$

$$+ cP_n(d-2)\, e^{2t} y + c'P_{n-1}(d-2)\, e^{2t} y + c''P_{n-2}(d-2)\, e^{2t} y + \ldots + c^{(n)} e^{2t} y$$

$+ \&c. = T$ (a function of e^t),

which is evidently of the form (where $f_0(d)$, $f_1(d)$, &c. denote rational and integral functions of d)

$$f_0(d)\,y + f_1(d)\,e^t y + f_2(d)\,e^{2t} y + \&c. = T;$$

and in this form only differential coefficients affected with constant multipliers enter. By this transformation, the object of which is to bring all the variable quantities under the symbols of operation, we can in several cases effect the solution of the equation, or its reduction to a simpler form.

72. Thus suppose the first member reduced to a single term

$$f_r(d)\,e^{rt} y = T, \quad \text{or } e^{rt} f_r(d+r)\,y = T, \quad \text{or } f_r(d+r)\,y = e^{-rt}T,$$

an ordinary linear equation with constant coefficients. This happens to the equation

$$(a+bx)^n \frac{d^n y}{dx^n} + A\,(a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \&c. + Ly = X,$$

which by replacing $a+bx$ by x is first transformed into

$$b^n x^n \frac{d^n y}{dx^n} + Ab^{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + Ly = X_1,$$

and then, making $x = e^t$, into

$$b^n P_n(d)\,y + Ab^{n-1} P_{n-1}(d)\,y + \dots + Ly = T,$$

which is of the form $f_0(d)\,y = T$.

Ex. $x^3 \dfrac{d^3 y}{dx^3} + ax^2 \dfrac{d^2 y}{dx^2} + bx \dfrac{dy}{dx} + cy = X;$

this becomes $\{d^3 + (a-3)\,d^2 + (-a+b+2)\,d + c\}\,y = X,$

if therefore $a=3$, $b=1$, $c=-1$, and $X = x^m$, $\dfrac{d^3 y}{dt^3} - y = e^{mt}$,

$$\text{and } y = \frac{e^{mt}}{m^3-1} + c_0 + \beta \cos\left(\frac{2\pi t}{3} + \alpha\right)$$

$$= \frac{x^m}{m^3-1} + c_0 + \beta \cos\left(\frac{2\pi}{3} \log x + \alpha\right).$$

Or if $a=-3$, $b=7$, $c=-8$, then $(d-2)^3 y = X$,

$$\text{and } y = e^{2t}\,(\int dt)^3\, e^{-2t} X = x^2 \left(\int dx\,\frac{1}{x}\right)^3 . \frac{X}{x^2},$$

because $\int dt$ prefixed to any subject is equivalent to $\int dx\,\frac{1}{x}$.

73. The meaning, in the above and similar cases, of such a notation as $\left(\int dx\,\frac{1}{x}\right)^3 u$ is that the complex operation $\int dx\,\frac{1}{x}$ (which consists in dividing the subject by x, and then integrating it with respect to x) is performed three times in succession; first upon u which gives a result u_1, then upon u_1 which gives a result u_2, and thirdly upon u_2. Thus to get the complementary function in the above result, we must make $X=0$; then

$$\int dx\,\frac{1}{x}\,0 = \int dx\,0 = C_0, \quad \int dx\,\frac{1}{x}\,C_0 = C_0 \log x + C_1,$$

$$\int dx\,\frac{1}{x}\,(C_0 \log x + C_1) = \tfrac{1}{2} C_0 (\log x)^2 + C_1 (\log x) + C_2;$$

and the complete value of y is

$$y = x^2 \left(\int dx\,\frac{1}{x}\right)^3 \frac{X}{x^2} + x^2 \{\tfrac{1}{2} C_0 (\log x)^2 + C_1 (\log x) + C_2\}.$$

Hence for $\quad x^3 \dfrac{d^3y}{dx^3} - 3x^2 \dfrac{d^2y}{dx^2} + 7x \dfrac{dy}{dx} - 8y = \dfrac{x^3 (1-x)}{(1+x)^3}$,

$$y = x^2 \log(1+x) + x^2 \{c_0 + c_1 \log x + c_2 (\log x)^2\}.$$

74. Next suppose the first member of the transformed equation to consist only of two terms

$$f_m(d)\, e^{mt} y + f_n(d)\, e^{nt} y = T,$$

$$\text{or } e^{mt} f_m(d+m)\, y + e^{mt} f_n(d+m)\, e^{(n-m)t} y = T;$$

which by dividing by the coefficient of y, is reduced to the form

$$y + \phi(d)\, e^{rt} y = T_1.$$

Let $v + \psi(d)\, e^{rt} v = T_2$ be an auxiliary equation whose solution is known for any function T_2 of e^t, as the value of its second mem-

ber, and with which we wish to make the proposed equation coincide. Assume $y = f(d)\, v$ then

$$f(d)\, v + \phi(d)\, e^{rt} f(d)\, v = T_1,$$

$$\text{or } v + \frac{\phi(d)}{f(d)} \cdot f(d-r)\, e^{rt} v = \frac{T_1}{f(d)};$$

and in order that this may coincide with the auxiliary equation we must have the two conditions

$$\frac{T_1}{f(d)} = T_2, \quad \frac{\phi(d)}{f(d)} \cdot f(d-r) = \psi(d),$$

the latter of which gives

$$f(d) = \frac{\phi(d)}{\psi(d)} f(d-r) = \frac{\phi(d)}{\psi(d)} \cdot \frac{\phi(d-r)\ldots}{\psi(d-r)\ldots} = P_r \frac{\phi(d)}{\psi(d)},$$

where P_r denotes the product of an infinite number of factors formed from $\dfrac{\phi(d)}{\psi(d)}$ by replacing d by $d-r$, $d-2r$, &c. Hence, $f(d)$ being known, $T_2 = \{f(d)\}^{-1} T_1$ can be computed; then v is known from $v + \psi(d)\, e^{rt} v = T_2$ an equation which is supposed to admit of being solved when T_2 is any given function of e^t, and then $y = P_r \dfrac{\phi(d)}{\psi(d)}\, v$.

Ex. 1. $$x^2 \frac{d^2y}{dx^2} + (a+1)\, x \frac{dy}{dx} + (b + q^2x^2)\, y = X;$$

$$\therefore\ d(d-1)\, y + (a+1)\, dy + (b + q^2 e^{2t})\, y = T,$$

$$\text{or } (d^2 + ad + b)\, y + q^2 e^{2t} y = T;$$

$$\therefore\ y + \frac{q^2}{d^2 + ad + b} e^{2t} y = T_1.$$

As the proposed equation can be solved when $a = -1$, $b = 0$, let the auxiliary equation be

$$\frac{d^2v}{dx^2} + q^2 v = X_2, \text{ or } v + \frac{q^2}{d^2 - d} e^{2t} v = T_2.$$

Assume $y = f(d)\, v$;

$$\therefore f(d)\, v + \frac{q^2}{d^2 + ad + b} e^{2t} f(d)\, v = T_1,$$

$$\text{or } v + \frac{q^2}{d^2+ad+b}\,\frac{f(d-2)}{f(d)}\,e^{2t}v = \frac{T_1}{f(d)};$$

$$\therefore\; f(d) = \frac{d^2-d}{d^2+ad+b}f(d-2) = P_2\frac{d^2-d}{d^2+ad+b};$$

and such values must be given to the constants a and b as will produce a value of $f(d)$ that will allow y to be determined from the equation $y = f(d)\,v = f(d)\,\beta\cos(qx+\alpha)$ when $X=0$.

If q^2 be preceded by a negative sign, the value to be used for v in this and the following examples will be $\alpha e^{qx}+\beta e^{-qx}$.

2. $x^2\dfrac{d^2y}{dx^2} - 2nx\dfrac{dy}{dx} + (2n+q^2x^2)\,y = 0;$

$$f(d) = P_2\frac{d}{d-2n} = d\,(d-2)\,\ldots\,(d-2n+2);$$

$$\therefore\; y = d\,(d-2)\,(d-4)\,\ldots\,(d-2n+2)\,\beta\cos(qx+\alpha);$$

but the symbol of operation $\dfrac{d}{dt} - r$ prefixed to any subject, is equivalent to

$$e^{rt}\frac{d}{dt}e^{-rt} = x^r.\,x\frac{d}{dx}\frac{1}{x^r} = x^{r+1}\frac{d}{dx}\frac{1}{x^r};$$

$$\therefore\; y = x\frac{d}{dx}\,.\,x^3\frac{d}{dx}\frac{1}{x^2}\,.\,x^5\frac{d}{dx}\frac{1}{x^4}\,\ldots\,x^{2n-1}\frac{d}{dx}\frac{v}{x^{2n-2}}$$

$$= \frac{1}{x^2}\left(x^3\frac{d}{dx}\right)^n\frac{\beta\cos(qx+\alpha)}{x^{2n-2}}.$$

Let $n=2$, then the equation is

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4+q^2x^2)\,y = 0,$$

and its solution

$$y = \beta\,\{-q^2x^2\cos(qx+\alpha) + qx\sin(qx+\alpha)\}.$$

3. $x^2\dfrac{d^2y}{dx^2} - 2nx\dfrac{dy}{dx} + q^2x^2y = 0;$

$$f(d) = P_2\frac{d-1}{d-2n-1} = (d-1)\,(d-3)\ldots(d-2n+1);$$

$$\therefore \; y = (d-1)(d-3) \ldots (d-2n+1)\, \beta \cos(qx+\alpha);$$

and, as shewn in the preceding example, $\dfrac{d}{dt} - r$ prefixed to any subject is equivalent to $x^{r+1} \dfrac{d}{dx} x^{-r}$;

$$\therefore \; y = x^2 \frac{d}{dx}\frac{1}{x} \cdot x^4 \frac{d}{dx}\frac{1}{x^3} \cdot x^6 \frac{d}{dx}\frac{1}{x^5} \ldots x^{2n} \frac{d}{dx}\frac{v}{x^{2n-1}}$$

$$= \frac{1}{x}\left(x^3 \frac{d}{dx}\right)^n \frac{\beta \cos(qx+\alpha)}{x^{2n-1}}.$$

4. $\quad x^2 \dfrac{d^2y}{dx^2} - n(n+1)\, y + q^2x^2y = 0.$

If in this equation we replace y by $x^{-n}y$, it is reduced to the preceding example; and consequently its solution is

$$y = \frac{1}{x^{n+1}}\left(x^3 \frac{d}{dx}\right)^n \frac{\beta \cos(qx+\alpha)}{x^{2n-1}}.$$

if we make n equal to 1, 2, we shall find the corresponding values of y to be

$$-\beta q \sin(qx+\alpha) - \frac{\beta}{x}\cos(qx+\alpha);$$

$$\text{and} - \beta\left(q^2 - \frac{3}{x^2}\right)\cos(qx+\alpha) + \frac{3\beta q}{x}\sin(qx+\alpha);$$

α and β denoting arbitrary constants.

5. $\quad x^2 \dfrac{d^2y}{dx^2} + (x + ax + qx^2)\dfrac{dy}{dx} + by = 0;$

$$\therefore \; y + \frac{d-1}{d^2+ad+b}\, qe^t y = 0.$$

Making $a = -1$, $b = 0$, let the auxiliary equation be

$$v + \frac{qe^t v}{d} = 0, \text{ where } v = \alpha + \beta e^{-qx},$$

$$\therefore \; f(d) = P_1 \frac{d^2 - d}{d^2 + ad + b};$$

and the constants a and b may receive suitable values as in the foregoing examples.

6. $x^2 \dfrac{d^2y}{dx^2} + qx^2 \dfrac{dy}{dx} - n(n+1)\,y = 0$, here $a = -1$, $b = -n\,(n+1)$,

$$\therefore\ f(d) = P_1 \frac{d\,(d-1)}{(d+n)\,(d-n-1)} = \frac{(d-1)\,(d-2)\,\ldots\,(d-n)}{(d+1)\,(d+2)\,\ldots\,(d+n)}.$$

But $d - r$ is equivalent to $x^{r+1} \dfrac{d}{dx} x^{-r}$, and $(d+r)^{-1}$ to $x^{-r}\,(\int dx)\,x^{r-1}$;

$$\therefore\ y = \left(x^2 \frac{d}{dx}\right)^n \frac{1}{x^{n-1}}\,(x^{-2} \int dx)^n\, x^{n-1}\,(\alpha + \beta e^{-qx}).$$

Let $n = 1$,

$$y = \left(x^2 \frac{d}{dx}\right) x^{-2} \int dx\,(\alpha + \beta e^{-qx}) = x^2 \frac{d}{dx}\,(\alpha x^{-1} - \frac{\beta}{q} x^{-2} e^{-qx} + Cx^{-2})$$

$$= -\alpha - \frac{2C}{x} + \beta \left(1 + \frac{2}{qx}\right) e^{-qx}. \quad \text{Hence, } y = -\alpha - \frac{2C}{x}$$

is a solution, and if we substitute it in the proposed equation to determine the supernumerary constant, we find $C = -\dfrac{\alpha}{q}$. The necessity of adding a constant in finding y arises from the disappearance of the factor $d - 1$ from the auxiliary equation.

$$\text{Let } n = 2,\quad y = \left(x^2 \frac{d}{dx}\right)^2 \frac{1}{x}\,(x^{-2} \int dx)^2\, x\,(\alpha + \beta e^{-qx})$$

$$= \left(x^2 \frac{d}{dx}\right)^2 \left(\frac{\alpha}{2x^2} + \frac{\beta}{q^2 x^4} e^{-qx} - \frac{C}{x^4} + \frac{C'}{x^3}\right)$$

$$= \alpha + \frac{6C'}{x} - \frac{12C}{x^2} + \beta \left(1 + \frac{6}{qx} + \frac{12}{q^2x^2}\right) e^{-qx}.$$

Since $y = \alpha + \dfrac{6C'}{x} - \dfrac{12C}{x^2}$ is a solution, the supernumerary constants C and C' must be determined by substituting this value in the proposed equation; and the exact solution will be found to be

$$y = \alpha \left(1 - \frac{6}{qx} + \frac{12}{q^2x^2}\right) + \beta \left(1 + \frac{6}{qx} + \frac{12}{q^2x^2}\right) e^{-qx}.$$

7. $(1-x^2)\,x^2\dfrac{d^2y}{dx^2}-(2m+x^2)\,x\dfrac{dy}{dx}+\{m\,(m+1)+q^2x^2\}\,y=0.$

$$\text{or } y-\frac{(d-2)^2-q^2}{(d-m)\,(d-m-1)}\,e^{2t}y=0.$$

Making $m=0$, let the auxiliary equation be

$$(1-x^2)\,\frac{d^2v}{dx^2}-x\,\frac{dv}{dx}+q^2v=0,$$

where $v=\alpha\cos(q\cos^{-1}x)+\beta\sin(q\cos^{-1}x)$, (Art. 69)

$$\text{or } v-\frac{(d-2)^2-q^2}{d\,(d-1)}\,e^{2t}v=0\,;$$

$$\therefore\ f(d)=P_2\,\frac{d\,(d-1)}{(d-m)\,(d-m-1)}=d\,(d-1)\,\ldots\,(d-m+1)\,;$$

$$\therefore\ y=d\,(d-1)\,(d-2)\,\ldots\,(d-m+1)\,v$$

$$=x^m\left(\frac{d}{dx}\right)^m\{\alpha\cos(q\cos^{-1}x)+\beta\sin(q\cos^{-1}x)\}.$$

If q^2 have a negative sign before it, the value of v is $\alpha e^{q\cos^{-1}x}+\beta e^{-q\cos^{-1}x}$.

75. Thirdly, suppose that the transformed equation can be put under the form

$$y+a_1f(d)\,e^ty+a_2f(d)f(d-1)\,e^{2t}y$$
$$+a_3f(d)f(d-1)f(d-2)\,e^{3t}y+\&\text{c.}=T;$$

then since $f(d-1)\,e^{2t}y=e^tf(d)\,e^ty$;

$$\therefore\ f(d)f(d-1)\,e^{2t}y=f(d)\,e^tf(d)\,e^ty=\{f(d)\,e^t\}^2y,$$

and generally

$$f(d)f(d-1)f(d-2)\,\ldots f(d-n+1)\,e^{nt}y=\{f(d)\,e^t\}^ny,$$

consequently the proposed equation becomes, if we denote the symbol of operation $f(d)\,e^t$ by ρ,

$$\{1+a_1\rho+a_2\rho^2+\ldots+a_m\rho^m\}\,y=T;$$

$$\therefore\ y=(1+a_1\rho+a_2\rho^2+\ldots+a_m\rho^m)^{-1}\,T$$

$$=\left(\frac{N_1}{1-q_1\rho}+\frac{N_2}{1-q_2\rho}+\ldots+\frac{N_m}{1-q_m\rho}\right)T,$$

resolving the function of ρ with which T is affected into its partial fractions,

$$\text{or } y = N_1 u_1 + N_2 u_2 + \dots + N_m u_m, \text{ where } u_1, u_2, \text{\&c.}$$

are determined from the equations,

$$u_1 - q_1 f(d)\, e^t u_1 = T_1, \quad u_2 - q_2 f(d)\, e^t u_2 = T_1, \text{ \&c.}$$

76. If $f(d) = d^{-1}$, the equation just treated of is the linear equation with constant coefficients:

$$x^n \frac{d^n y}{dx^n} + p_1 x\, x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \text{\&c.} = x^n X;$$

for this leads to

$$P_n(d)\, y + p_1 P_{n-1}(d-1)\, e^t y + p_2 P_{n-2}(d-2)\, e^{2t} y + \text{\&c.} = T;$$

$$\text{or } y + \frac{p_1}{d} e^t y + \frac{p_2}{d(d-1)} e^{2t} y + \text{\&c.} = \frac{T}{P_n(d)};$$

which is of the assumed form. For all other values of $f(d)$ the equation has variable coefficients. If the equation be

$$x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \text{\&c.} = X;$$

$$f(d) = \frac{1}{d - n + 1},$$

which leads to a linear equation of the first order for the determination of u_1, u_2, &c.

$$x^n \frac{d}{dx}\left(\frac{u_1 - T_1}{x^{n-1}}\right) = q_1 x u_1; \text{ if } T_1 = 0, \text{ this gives } u_1 = C x^{n-1} e^{q_1 x}.$$

If $f(d) = a + b\,(d - r)^{-1}$, the particular values of y are determined from linear equations of the first order,

$$\frac{d}{dx}\{u_1 (1 - q_1 a x)\, x^{-r} - T_1 x^{-r}\} = q_1 b u_1 x^{-r}.$$

Ex. $$(1 + ax + bx^2)\, x^2 \frac{d^2 y}{dx^2} - \{2r + (r-2)\, ax - 4bx^2\}\, x \frac{dy}{dx}$$
$$+ \{r(r+1) - rax + 2bx^2\}\, y = X.$$

Changing the independent variable into $t = \log x$, this becomes

$$(1 + ae^t + be^{2t})\, d\,(d-1)\, y - \{2r + (r-2)\, ae^t - 4be^{2t}\}\, dy$$
$$+ \{r\,(r+1) - rae^t + 2be^{2t}\}\, y = T,$$

$$\text{or}\ \ (d-r)\,(d-r-1)\, y + ae^t\,(d+1)\,(d+1-r-1)\, y$$
$$+ be^{2t}\,(d+2)\,(d+2-1)\, y = T,$$

$$\text{or}\ \ (d-r)\,(d-r-1)\, y + ad\,(d-r-1)\, e^t y + bd\,(d-1)\, e^{2t} y = T,$$

$$\text{or}\ \ y + a\frac{d}{d-r}\, e^t y + b\frac{d}{d-r}\cdot\frac{d-1}{d-r-1}\, e^{2t} y = T_1,$$

$$\text{or}\ \ (1 + a\rho + b\rho^2)\, y = T_1,\ \text{putting}\ \frac{d}{d-r}\, e^t = \rho,$$

$$\text{or}\ \ (1 + a\rho + b\rho^2)\, y = (1 - q_1\rho)\,(1 - q_2\rho)\, y = T_1\,;$$

$$\therefore\ y = \left(\frac{N_1}{1 - q_1\rho} + \frac{N_2}{1 - q_2\rho}\right) T_1 = N_1 u_1 + N_2 u_2,$$

where u_1 is given by the equation

$$(1 - q_1\rho)\, u_1 = T_1, \quad \text{or}\ u_1 - q_1\frac{d}{d-r}\, e^t u_1 = T_1,$$

$$\text{or}\ \ (d-r)\,(u_1 - T_1) = q_1 de^t u_1,$$

$$\text{or}\ \ x^{r+1}\frac{d}{dx}\left(\frac{u_1 - T_1}{x^r}\right) = q_1 x\frac{d}{dx}\,(xu_1).$$

Further applications of this method may be seen in the *Philosophical Transactions*, where Mr Boole has treated this subject with great generality. As a concluding application of the method by changing the independent variable, let us solve the equation

$$(x^2 - 4)\frac{d^2y}{dx^2} + 2\,(x+1)\frac{dy}{dx} - n\,(n+1)\, y = X.$$

Taking z for independent variable where $z + \dfrac{1}{z} = x$, we find

$$(z-1)\, z^2\frac{d^2y}{dz^2} + 2z^2\frac{dy}{dx} - n\,(n+1)\,(z-1)\, y = Z.$$

Now assume $z = e^t$, then

$$(e^t - 1)\, d\,(d-1)\, y + 2e^t dy - n\,(n+1)\,(e^t - 1)\, y = Z,$$

$$\text{or}\ \ e^t\,(d+n+1)\,(d-n)\, y - (d-n-1)\,(d+n)\, y = Z,$$

$$\text{or}\ \ (d+n)\,(d-n-1)\,(e^t y - y) = Z;$$

$$\therefore\ \ (e^t - 1)\, y = \frac{1}{2n+1}\left(\frac{1}{d-n-1} - \frac{1}{d+n}\right) Z,$$

$$\text{or}\ \ (2n+1)\,(z-1)\, y = z^{n+1} \int dz\, z^{-n-2}\, Z - z^{-n} \int dz\, z^{n-1}\, Z.$$

If $Z = 0$, we have

$$y = \frac{cz^{n+1} + c'z^{-n}}{z-1};$$

and if for z we substitute its value $\frac{1}{2}(x + \sqrt{x^2 - 4})$, we shall find

$$y = cu + c' \sqrt{\frac{x+2}{x-2}} \left\{u + 2\,(x-2)\,\frac{du}{dx}\right\},$$

where $u = x^n + x^{n-1} - (n-1)\, x^{n-2} - (n-2)\, x^{n-3} + \text{\&c.}$

the general term of the series for u being

$$(-1)^r \frac{(n-r-1)\,(n-r-2)\,\ldots\,(n-2r+1)}{1\,.\,2\,.\,3\,\ldots\,r}\,(n-2r+nx-rx)\, x^{n-2r-1}.$$

Simultaneous Equations.

In these equations, which sometimes occur in the higher parts of Dynamics, instead of one equation between x, y, and the differential coefficients of y with respect to x, being given to determine the relation between x and y; we have two equations containing x, y, and t (of which x and y are considered as functions) and the differential coefficients of x and y relative to t, to find that relation.

77. To integrate the simultaneous equations of the first order,

$$ax + by + \frac{dx}{dt} = T, \qquad a'x + b'y + \frac{dy}{dt} = T',$$

T and T' denoting functions of t. Multiplying the latter by an indeterminate quantity m, and adding it to the former, we get

$$ax + by + m\,(a'x + b'y) + \frac{d}{dt}\,(x + my) = T + mT',$$

$$\text{or}\ \ \frac{d}{dt}\,(x + my) + (a + ma')\,(x + \frac{b + mb'}{a + ma'}\,y) = T + mT'.$$

Let $\dfrac{b + mb'}{a + ma'} = m$, which will give two values of m, m_1 and m_2; then the equation, under this condition, becomes a linear equation of the first order; and we obtain by integration

$$(x + my)\,e^{(a+ma')t} = \int dt\,(T + mT')\,e^{(a+ma')t};$$

and by substituting successively the two values of m, we obtain two primitive equations each containing an arbitrary constant, which will furnish values of x and y in terms of t, and the relation between x and y, if t be eliminated. If the two values of m are equal, we shall obtain only one equation between x, y, and t; but if this can be solved with respect to x or y, and we substitute the value so found in one of the given equations, we shall obtain a second relation either between x and t, or between y and t; and then t may be eliminated as before.

Ex. 1. $\quad 5x - 2y + \dfrac{dx}{dt} = e^t, \quad 6y - x + \dfrac{dy}{dt} = e^{2t};$

$$\therefore\ (5 - m)\,x + (-2 + 6m)\,y + \frac{d}{dt}\,(x + my) = e^t + me^{2t},$$

$$\text{let}\ \frac{-2 + 6m}{5 - m} = m,\ \text{or}\ m^2 + m - 2 = 0,\ \text{or}\ m = 1,\ \text{or}\ -2;$$

$$\therefore\ \frac{d}{dt}\,(x + y) + 4\,(x + y) = e^t + e^{2t};$$

$$\therefore\ x + y = \frac{e^t}{5} + \frac{e^{2t}}{6} + Ce^{-4t};$$

$$\text{similarly}\ x - 2y = \frac{e^t}{8} - \frac{2}{9}e^{2t} + C_1e^{-7t},$$

which determine x and y in terms of t.

Ex. 2. $\dfrac{dx}{dt} + 5x + y = e^t, \quad \dfrac{dy}{dt} + 3y - x = e^{2t};$

$$\therefore \frac{d}{dt}(x + my) + (5 - m)x + (1 + 3m)y = e^t + me^{2t};$$

$$\therefore \frac{1 + 3m}{5 - m} = m, \quad \text{or } 1 - 2m + m^2 = (1 - m)^2 = 0.$$

Hence the values of m are each $= 1$, and integrating, we find

$$x + y = C_1 e^{-4t} + \frac{1}{5}e^t + \frac{1}{6}e^{2t}.$$

By means of this, eliminate y from the first equation, and we get

$$\frac{dx}{dt} + 4x = \frac{4}{5}e^t - \frac{e^{2t}}{6} - C_1 e^{-4t};$$

$$\therefore x = \frac{4}{25}e^t - \frac{1}{36}e^{2t} - C_1 te^{-4t} + C_2 e^{-4t},$$

$$\text{and } y = \frac{1}{25}e^t + \frac{7}{36}e^{2t} + C_1(1 + t)e^{-4t} - C_2 e^{-4t}.$$

The more general form

$$ax + by + A\frac{dx}{dt} + B\frac{dy}{dt} = T, \quad a'x + b'y + A'\frac{dx}{dt} + B'\frac{dy}{dt} = T',$$

may evidently be reduced to the above by successively eliminating $\dfrac{dy}{dt}, \dfrac{dx}{dt}$.

78. To integrate the simultaneous equations of the second order,

$$ax + by + c + \frac{d^2x}{dt^2} = 0, \quad a'x + b'y + c' + \frac{d^2y}{dt^2} = 0.$$

Multiplying the latter by an indeterminate quantity m, and adding it to the former, we get

$$(a + ma')x + (b + mb')y + c + mc' + \frac{d^2}{dt^2}(x + my) = 0,$$

$$\text{or } \frac{d^2}{dt^2}(x+my+c_1)+(a+ma')(x+my+c_1)=0 \quad (1),$$

$$\text{if } \frac{b+mb'}{a+ma'}=m \quad (2), \qquad \frac{c+mc'}{a+ma'}=c_1;$$

therefore, integrating equation (1), and substituting successively the two values of m given by equation (2), we obtain the two required primitives; or if the values of m be equal, we must proceed as in the former case.

Ex. 1. $\dfrac{d^2x}{dt^2}-(3x+4y-3)=0, \quad \dfrac{d^2y}{dt^2}+(x+y+5)=0;$

$$\therefore \frac{d^2}{dt^2}(x+my)+(m-3)\left(x+\frac{m-4}{m-3}y+\frac{3+5m}{m-3}\right)=0.$$

$$\text{Let } \frac{m-4}{m-3}=m, \quad \text{or } m^2-4m+4=(m-2)^2=0;$$

$$\therefore \frac{d^2}{dt^2}(x+2y)-(x+2y-13)=0;$$

$$\therefore x+2y-13=ce^t+c'e^{-t},$$

and eliminating x from the latter of the given equations, we find

$$\frac{d^2y}{dt^2}-y+18+ce^t+c'e^{-t}=0;$$

$$\therefore y=18-\frac{c}{2}(t-\tfrac{1}{2})e^t+\frac{c'}{2}te^{-t}+ae^t+a'e^{-t},$$

$$\text{and } x=-23+c(t+\tfrac{1}{2})e^t-c'(t-1)e^{-t}-2ae^t-2a'e^{-t}.$$

Ex. 2. $\dfrac{d^2x}{dt^2}+a\dfrac{dx}{dt}=0, \quad \dfrac{d^2y}{dt^2}+a\dfrac{dy}{dt}+b=0.$

79. If we have three variables x, y, z, which are functions of t, and if

$$\frac{dx}{dt}+ax+by+cz=T,$$

$$\frac{dy}{dt}+a_1x+b_1y+c_1z=T_1,$$

$$\frac{dz}{dt}+a_2x+b_2y+c_2z=T_2,$$

then if we multiply the second and third by indeterminate constants m, m', and add them to the first, and assume

$$a + ma_1 + m'a_2 = s,$$

$$b + mb_1 + m'b_2 = ms, \quad c + mc_1 + m'c_2 = m's,$$

we get

$$\frac{d}{dt}(x + my + m'z) + s(x + my + m'z) = T + mT_1 + m'T_2.$$

But the three preceding equations give

$$s - a = ma_1 + m'a_2, \quad m(s - b_1) = m'b_2 + b, \quad m'(s - c_2) = mc_1 + c;$$

and if values of m and m' be obtained from the two latter and substituted in the former, we get to determine s the cubic equation

$$(s-a)(s-b_1)(s-c_2) - b_2c_1(s-a) - a_2c(s-b_1) - ba_1(s-c_2) - a_1b_2c - a_2bc_1 = 0.$$

Hence if m_1, m_1'; m_2, m_2'; m_3, m_3'; be the values of m, m', corresponding to the three roots of this cubic, we have by solving the linear equations,

$$x + m_1y + m_1'z = F_1(t),$$
$$x + m_2y + m_2'z = F_2(t),$$
$$x + m_3y + m_3'z = F_3(t),$$

from which equations x, y, z may be found in terms of t. It may be observed that if $b_2 = c_1$, $c = a_2$, and $a_1 = b$, so that the cubic equation is

$$(s - a)(s - b_1)(s - c_2) - c_1^2(s - a) - a_2^2(s - b_1) - b^2(s - c_2) - 2c_1a_2b = 0,$$

then all its roots are real (*Theory of Algeb. Eq.* Art. 58).

80. This method readily leads to Jacobi's solution of the equation

$$(A + A'x + A''y)\left(x\frac{dy}{dx} - y\right) - (B + B'x + B''y)\frac{dy}{dx} + (C + C'x + C''y) = 0,$$

which, upon introducing the independent variable t, becomes

$$\{(A + A'x + A''y)x - (B + B'x + B''y)\}\frac{dy}{dt} = \{(A + A'x + A''y)y - (C + C'x + C''y)\}\frac{dx}{dt},$$

and resolves itself into

$$\frac{dx}{dt} - (A + A'x + A''y)\,x + (B + B'x + B''y) = 0,$$

$$\frac{dy}{dt} - (A + A'x + A''y)\,y + (C + C'x + C''y) = 0\,;$$

or $\{$putting $\frac{dz}{dt} = -(A + A'x + A''y)\,z\}$ into the system,

$$\frac{dz}{dt} + Az + A'xz + A''yz = 0,$$

$$\frac{d\,(xz)}{dt} + Bz + B'xz + B''yz = 0,$$

$$\frac{d\,(yz)}{dt} + Cz + C'xz + C''yz = 0.$$

Multiply the second and third by m and m' respectively, and add them to the first, and assume

$$A + mB + m'C = s,\quad A' + mB' + m'C' = ms,\quad A'' + mB'' + m'C'' = m's \quad (1);$$

$$\text{then } \frac{d}{dt}(z + mxz + m'yz) + s\,(z + mxz + m'yz) = 0\,;$$

which gives $z\,(1 + mx + m'y)\,e^{st} = C$, or $zue^{st} = C$.

But equations (1) lead successively to

$$s - A = mB + m'C,\quad m(s - B') = A' + m'C',\quad m'(s - C'') = A'' + mB'';$$

the two latter of which determine m and m' from s by the equations

$$\left.\begin{aligned} m\,\{(s - B')\,(s - C'') - B''C'\} &= A'\,(s - C'') + C'A'' \\ m'\,\{(s - C'')\,(s - B') - B''C'\} &= A''\,(s - B') + A'B'' \end{aligned}\right\} \quad (2),$$

and the values thence obtained of m and m' substituted in the former give for finding s the cubic equation

$$(s - A)\,(s - B')\,(s - C'') - B''C'\,(s - A) - A''C\,(s - B')$$
$$- A'B\,(s - C'') + A''BC' + A'B''C = 0.$$

Let the three values of s in this equation be s_1, s_2, s_3; and let the corresponding values of m and m', be m_1 and m'_1, m_2 and m'_2, m_3 and m'_3; then corresponding to the three roots of the cubic we get the three integrals.

$$zu_1e^{s_1t} = C_1, \quad zu_2e^{s_2t} = C_2, \quad zu_3e^{s_3t} = C_3,$$

and if we raise these equations respectively to the powers $s_2 - s_3$, $s_3 - s_1$, $s_1 - s_2$, and take their product, z and e^t are eliminated, and we find the required solution

$$u_1^{s_2-s_3} \times u_2^{s_3-s_1} \times u_3^{s_1-s_2} = C;$$

$$\text{or} \quad (1 + m_1x + m'_1y)^{s_2-s_3} \times (1 + m_2x + m'_2y)^{s_3-s_1}$$
$$\times (1 + m_3x + m'_3y)^{s_1-s_2} = C,$$

where m_1, m'_1 are given in terms of s_1, m_2, m'_2 in terms of s_2, and m_3, m'_3 in terms of s_3, by equations (2). This includes as a particular case the equation (Ex. 7, Art. 18), in which Euler separated the variables,

$$(a + bx + cx^2 + y)\frac{dy}{dx} - (n + cx)\,y = 0,$$

$$\text{or} \quad cx\left(x\frac{dy}{dx} - y\right) + (a + bx + y)\frac{dy}{dx} - ny = 0.$$

81. When the proposed equations are linear with constant coefficients, we may separate the symbols of operation from those of quantity and then obtain by the ordinary processes of elimination an equation containing only one of the unknown quantities: for as the symbols of operation here employed combine according to the same laws as ordinary algebraical quantities, they may be treated precisely as if they were symbols of quantity. Thus the equations of Art. 77 $\left(\text{suppressing the } dt \text{ in } \dfrac{d}{dt}\right)$ may be written

$$(d + a)\,x + by = T, \quad (d + b')\,y + a'x = T';$$

the latter gives $y = \dfrac{T' - a'x}{d + b'}$; and substituting this value in the former we get

$$(d + a)\,(d + b')\,x - a'bx = (d + b')\;T - bT',$$

an ordinary linear equation with constant coefficients, from which the value of x may be obtained, containing two arbitrary constants; and thence the value of y by the equation

$$y = \frac{1}{b}\,T - \frac{1}{b}\,(d + a)\,x.$$

Ex. $\dfrac{dx}{dt}+2cx-y=t^2,\quad \dfrac{dy}{dt}-2cy+5c^2x=e^{mt}$;

the latter gives for y the value $y=\dfrac{e^{mt}-5c^2x}{d-2c}$,

and substituting this in the former we get

$$(d^2+c^2)\,x=e^{mt}+2\,(t-ct^2).$$

$$\therefore\ x=\frac{e^{mt}}{m^2+c^2}+\frac{2}{c^2}\,(t-ct^2)+\frac{4}{c^3}+\beta\cos\,(ct+\alpha)\,;$$

and $y=(d+2c)\,x-t^2$

$$=\frac{m+2c}{m^2+c^2}e^{mt}-5t^2+\frac{10}{c^2}-\beta c\sin\,(ct+\alpha)+2\beta c\cos\,(ct+\alpha).$$

82. Again, suppose the equations of Art. 78 to be

$$(d^2+ad+b)\,x=(md+n)\,y+T,$$
$$(d^2-ad+b')\,y=(m'd+n')\,x+T'\,;$$

obtaining a value of y from the latter, substituting it in the former and reducing, we find

$$\{d^4-(a^2-b-b'+mm')\,d^2+(ab'-ab-mn'-m'n)\,d+bb'-nn'\}\,x$$
$$=(d^2-ad+b')\,T+(md+n)\,T'\,;$$

from which the value of x may be obtained; and if it be substituted in one of the proposed equations, we may deduce the value of y.

Ex. 1. $(d-1)^2\,x=2dy+\cos t$

$$(d^2+2d+6)\,y=-\,(d+5)\,x+\sin t\,;$$

eliminating y, we find

$$(d^4+5d^2+6)\,x=7\cos t-2\sin t,$$

$$\therefore\ x=\frac{7}{2}\cos t-\sin t+c\cos\,(t\sqrt{2}+\alpha)+c'\cos\,(t\sqrt{3}+\beta)\,;$$

and the value of y may be now obtained from the former of the proposed equations.

Ex. 2. $\{d^2 + \left(n + \frac{1}{n}\right) d + n^2\} x = dy + T,$

$$\left\{d^2 - \left(n + \frac{1}{n}\right) d + \frac{1}{n^2}\right\} y = \left\{m^2 d - \left(n + \frac{1}{n}\right)\left(n^2 - \frac{1}{n^2}\right)\right\} x,$$

eliminating y, we find

$$\{d^4 - (2 + m^2) d^2 + 1\} x = \left\{d^2 - \left(n + \frac{1}{n}\right) d + \frac{1}{n^2}\right\} T,$$

or $(d^2 - md - 1)(d^2 + md - 1) x = \left\{d^2 - \left(n + \frac{1}{n}\right) d + \frac{1}{n^2}\right\} T,$

which gives x; and then y can be obtained from the former of the given equations. In these examples d is used for $\frac{d}{dt}$.

Solutions expressed by Definite Integrals.

83. Sometimes the complete integral of a differential equation may be expressed by a definite integral, as in the following instances.

Ex. 1. $\frac{d^{n-1} y}{dx^{n-1}} = xy + m.$

Let $v_1 = \alpha_1 \int_0^\infty dt e^{\alpha_1 tx} . e^{-\frac{t^n}{n}}$, α_1 being a constant quantity;

$$\text{then } \frac{d^{n-1}}{dx^{n-1}} v_1 = \alpha_1^n \int_0^\infty dt\, t^{n-1} e^{\alpha_1 tx} e^{-\frac{t^n}{n}} = \alpha_1^n (1 + xv_1);$$

change α_1 into α_2, α_3, &c., and let v_2, v_3, &c. be the corresponding values of v_1,

$$\therefore \frac{d^{n-1}}{dx^{n-1}} v_2 = \alpha_2^n (1 + xv_2),$$

....

$$\frac{d^{n-1}}{dx^{n-1}} v_n = \alpha_n^n (1 + xv_n).$$

Now suppose $\alpha_1, \alpha_2, \dots \alpha_n$ to be the n^{th} roots of unity; then multiplying each of these equations by arbitrary constants

c_1, c_2, &c. such that $c_1 + c_2 + \dots + c_n = m$, and adding them together, we find

$$\frac{d^{n-1}}{dx^{n-1}}(c_1v_1 + c_2v_2 + \dots + c_nv_n) = m + x(c_1v_1 + c_2v_2 + \dots + c_nv_n).$$

Hence $y = c_1v_1 + c_2v_2 + \dots + c_nv_n$ is the complete integral of the proposed equation, containing only $n-1$ arbitrary constants by reason of the equation of condition; or it may be written

$$y = \int_0^\infty dt\, e^{-\frac{t^n}{n}}\{ce^{tx} + c_1\alpha e^{\alpha tx} + c_2\alpha^2 e^{\alpha^2 tx} + \dots + c_{n-1}\alpha^{n-1}e^{\alpha^{n-1}tx}\},$$

α being a primitive root of $k^n - 1 = 0$.

Let $m = 0$, $n = 3$, then $\dfrac{d^2y}{dx^2} = xy$, the solution of which is

$$y = \int_0^\infty dt\, e^{-\frac{1}{3}t^3 - \frac{1}{2}xt}\left\{Be^{\frac{3tx}{2}} + \tfrac{1}{2}(B + A\sqrt{3})\cos\frac{\sqrt{3}tx}{2}\right.$$
$$\left. - \tfrac{1}{2}(A + B\sqrt{3})\sin\frac{\sqrt{3}tx}{2}\right\}.$$

Let $m = 0$, $n = 2$, then $\dfrac{dy}{dx} = xy$, and

$$y = C\int_0^\infty dt\, e^{-\frac{t^2}{2}}(e^{xt} + e^{-xt}) = \frac{2C}{\sqrt{-1}}\int_0^\infty dz\, e^{\frac{z^2}{2}}\cos xz, \text{ putting } t = z\sqrt{-1},$$

$$= \frac{2C}{\sqrt{-1}} \cdot \frac{\sqrt{\pi}e^{\frac{x^2}{2}}}{\sqrt{-2}} = C\sqrt{2\pi}e^{\frac{x^2}{2}}, \quad \text{(Integ. Cal. Art. 112).}$$

which agrees with the result obtained by direct integration.

Ex. 2. If the more general form were proposed,

$$\frac{d^{n-1}y}{dx^{n-1}} = (\alpha + \beta x)\, y,$$

assume $\alpha + \beta x = \beta^{\frac{n-1}{n}}t$; $\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \beta^{\frac{1}{n}}\dfrac{dy}{dt}$,

and $\dfrac{d^{n-1}y}{dx^{n-1}} = \beta^{\frac{n-1}{n}}\dfrac{d^{n-1}y}{dt^{n-1}} = \beta^{\frac{n-1}{n}}ty$; $\therefore \dfrac{d^{n-1}y}{dt^{n-1}} = ty$;

and to this form may the still more general case

$$\frac{d^{n-1}y}{dx^{n-1}} = axy + bx + cy + e$$

be reduced. (The above solution is taken from Crelle's Journal, Vol. x.)

Approximate Solutions of Differential Equations.

84. When all the known methods of integrating a proposed differential equation fail, we must endeavour to resolve it approximately, that is, to obtain from it the value of y in terms of x, in the form of a series. The first mode which presents itself of effecting this, is to assume for y a series arranged according to powers of x, with both its coefficients and exponents undetermined; for in most cases it happens that the exponents do not follow the progression of the natural numbers, and that particular artifices are requisite for discovering their law. When the form of this series is known, we may determine its coefficients by substituting it and its differential coefficients, for y, $\frac{dy}{dx}$, &c. in the proposed equation. The following application of the method to Riccati's Equation will give an idea of the mode of obtaining both the exponents and the coefficients.

Ex. 1. $\frac{dy}{dx} + by^2 - ax^n = 0$,

let $by = \frac{1}{z}\frac{dz}{dx}$; $\therefore \frac{d^2z}{dx^2} - abzx^n = 0$, or putting c for $-ab$,

$$\frac{d^2z}{dx^2} + czx^n = 0.$$

Assume $z = x^\alpha (A + Bx^\beta + Cx^{2\beta} + \text{\&c.})$; then

$$\left.\begin{array}{r} \alpha(\alpha-1)Ax^{\alpha-2} \\ +(\alpha+\beta)(\alpha+\beta-1)Bx^{\alpha+\beta-2} + (\alpha+2\beta)(\alpha+2\beta-1)Cx^{\alpha+2\beta-2} + \text{\&c.} \\ +cAx^{\alpha+n} \qquad\qquad\qquad + cBx^{\alpha+\beta+n} + \text{\&c.} \end{array}\right\} = 0.$$

Hence β must equal $n+2$; and then to determine α, A, B, &c., we have

$$\alpha(\alpha-1)A=0,$$

$$(\alpha+\ \ n+2)(\alpha+\ \ n+1)B+cA=0,$$

$$(\alpha+2n+4)(\alpha+2n+3)\ C+cB=0,$$

$$(\alpha+3n+6)(\alpha+3n+5)D+cC=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence α may be either zero or unity, and A remains undetermined; calling therefore A and A' the two values of A corresponding to $\alpha=0$, $\alpha=1$, we get

$$z=A\left\{1-\frac{cx^{n+2}}{(n+1)(n+2)}+\frac{c^2x^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)}-\&c.\right\}$$

$$+A'x\left\{1-\frac{cx^{n+2}}{(n+2)(n+3)}+\frac{c^2x^{2n+4}}{(n+2)(n+3)(2n+4)(2n+5)}-\&c.\right\}$$

and substituting this value of z in the expression $y=\dfrac{1}{bz}\dfrac{dz}{dx}$, we shall obtain the value of y, involving only one arbitrary constant $\dfrac{A}{A'}$.

As the terms of the above series have divisors of the forms $(n+2)i\mp 1$, where i is an integer; if n be such that

$$(n+2)i\mp 1=0,$$

one or other of the series will be illusory, and we shall only obtain a particular value of z; and if $n+2=0$, both series become infinite, but in that case the equation may be exactly integrated by Art. 16.

Ex. 2. $(x^2-4)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-n^2y=0.$

Assume $y=x^\alpha(a_0+a_1x^\beta+a_2x^{2\beta}+a_3x^{3\beta}+\&c.)$,

$$\therefore\ y=(\alpha^2-n^2)a_0x^\alpha+\{(\alpha+\beta)^2-n^2\}a_1x^{\alpha+\beta}-4\alpha(\alpha-1)a_0x^{\alpha-2}+\&c.$$

$$+\{(\alpha+r\beta)^2-n^2\}a_rx^{\alpha+r\beta}-4(\alpha-2r+2)(\alpha-2r+1)a_{r-1}x^{\alpha-2r}+\&c.,$$

the first term shews that a may be either $+n$ or $-n$, a_0 being the arbitrary constant; the next terms shew that $\beta = -2$, i.e. the series is a descending one; and that the coefficients are derived from one another by the law (taking $a = n$),

$$a_r = -\frac{(n-2r+2)(n-2r+1)}{r(n-r)} a_{r-1};$$

$$\therefore\ y = a_0 \{x^n - nx^{n-2} + \frac{n(n-3)}{1.2} x^{n-4} - \&c.\}$$

$$+ a'_0 \{x^{-n} + nx^{-n-2} + \frac{n(n+3)}{1.2} x^{-n-4} + \&c.\}.$$

As the solution of the proposed equation is (Art. 48)

$$y = a_0 \cos\left(n \cos^{-1}\frac{x}{2}\right) + a_0' \sin\left(n \cos^{-1}\frac{x}{2}\right),$$

if we make $x = 2\cos\theta$, we get the well-known result,

$$\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{1.2}(2\cos\theta)^{n-4} - \&c.$$

85. The transformation of Art. 71 may be employed in the case of linear equations to find the solution in the form of a series: for suppose the transformed equation to consist of three terms

$$f_0(d)y + f_1(d)e^t y + f_2(d)e^{2t}y = 0.$$

Let $y = u_m x^m + u_{m+1}x^{m+1} + \ldots + u_{r-2}x^{r-2} + u_{r-1}x^{r-1} + u_r x^r + \&c.$,

where u_r in the general term $u_r x^r$ is an invariable function of the index r, and x^m is the lowest power of x that can enter into the development of y. Now substitute this value of y (having first replaced x by e^t) in the proposed equation: then since

$$f(d)e^{rt} = f(r)e^{rt},$$

the first member will become

$$f_0(m)u_m e^{mt} + \{f_0(m+1)u_{m+1} + f_1(m+1)u_m\} e^{(m+1)t} + \ldots$$

$$+ \{f_0(r)u_r + f_1(r)u_{r-1} + f_2(r)u_{r-2}\} e^{rt} + \&c.$$

which will vanish, provided

$$f_0(m)\,u_m = 0, \quad f_0(m+1)\,u_{m+1} + f_1(m+1)\,u_m = 0,$$

and if for every value of r from $m+2$ upwards, we have

$$f_0(r)\,u_r + f_1(r)\,u_{r-1} + f_2(r)\,u_{r-2} = 0.$$

The first condition $f_0(m) = 0$, determines the values of m, u_m being an arbitrary constant: the second condition gives u_{m+1} in terms of u_m; and the third determines any coefficient u_r from the values of the two preceding. The third condition comprises the other two when we make $r = m$, $r = m+1$, if we consider that as $u_m x^m$ is the lowest power of x that can enter, $u_{m-1} = 0$, $u_{m-2} = 0$. As many real unequal values as exist for m, so many ascending series there are for y: if m have equal or imaginary values, the method fails.

Ex. 1. $$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0,$$

$$\text{or } d(d-1)\,y + \{n^2 - (d-2)^2\}\,e^{2t}y = 0.$$

Assume $y = u_m x^m + u_{m+1}x^{m+1} + \ldots + u_r x^r + \&c.$; then the first member of the equation becomes

$$m(m-1)\,u_m e^{mt} + m(m+1)\,u_{m+1}e^{(m+1)t} + \ldots$$
$$+ [r(r-1)\,u_r + \{n^2 - (r-2)^2\}\,u_{r-2}]\,e^{rt} + \&c.$$

which will vanish, provided we have

$$m(m-1)\,u_m = 0, \quad u_{m+1} = 0;$$

$$\text{and } u_r = -\frac{n^2 - (r-2)^2}{r(r-1)}\,u_{r-2}$$

for all values of r from $m+2$ upwards. Hence since the second term vanishes, the 4th, 6th, &c. term vanishes; and taking m successively equal to 0, and to 1, we get the complete value of y;

$$y = C\left(1 - \frac{n^2x^2}{1\,.\,2} + \frac{n^2(n^2 - 2^2)}{1\,.\,2\,.\,3\,.\,4}\,x^4 - \&c.\right)$$

$$+ C'\left\{x - \frac{n^2 - 1}{2\,.\,3}\,x^3 + \frac{(n^2-1)(n^2-3^2)}{2\,.\,3\,.\,4\,.\,5}\,x^5 - \&c.\right\},$$

which agrees with well-known results (Trig. Art. 134), as the solution of the equation is

$$y = C \cos (n \sin^{-1} x) + C' \sin (n \sin^{-1} x).$$

Ex. 2. $(x^2-4)\dfrac{d^2y}{dx^2}+2(x+1)\dfrac{dy}{dx}-n(n+1)\,y=0$;

$$\therefore\ e^{2t}(d+n+1)(d-n)\,y+2e^t dy-4d(d-1)\,y=0,$$

and dividing by e^{2t} in order to obtain y in a descending series, we get

$$(d+n+1)(d-n)\,y+2e^{-t}dy-4e^{-2t}d(d-1)\,y=0,$$

or $(d-n)(d+n+1)\,y+2(d+1)\,e^{-t}y-4(d+1)(d+2)\,e^{-2t}y=0.$

Since the values of m are n and $-(n+1)$, we may assume

$$y=u_n e^{nt}+u_{n-1}e^{(n-1)t}+u_{n-2}e^{(n-2)t}+u_{n-3}e^{(n-3)t}+\&c.$$

then the first member of the equation becomes

$$-2nu_{n-1}e^{(n-1)t}-2(2n-1)\,u_{n-2}e^{(n-2)t}-3(2n-2)\,u_{n-3}e^{(n-3)t}+\&c.$$

$$+2nu_n e^{(n-1)t}+(2n-2)\,u_{n-1}e^{(n-2)t}+(2n-4)\,u_{n-2}e^{(n-3)t}+\&c.$$

$$-4(n-1)\,nu_n e^{(n-2)t}-4(n-2)(n-1)\,u_{n-1}e^{(n-3)t}+\&c.$$

Hence, equating the coefficients of the powers of e^t to zero, we get

$$u_{n-1}=u_n,\quad u_{n-2}=-(n-1)\,u_n,\quad u_{n-3}=-(n-2)\,u_n,\ \&c.;$$

and since $u_n=c$, an arbitrary constant,

$$\frac{y}{c}=x^n+x^{n-1}-(n-1)\,x^{n-2}-(n-2)\,x^{n-3}+\dots$$

$$+\frac{(-1)^r}{\lfloor r}(n-r-1)(n-r-2)\dots(n-2r+1)\{n-2r+(n-r)\,x\}\,x^{n-2r-1}+\&c.$$

Similarly the series corresponding to $m=-n-1$ may be found.

86. In the preceding instances we arrive immediately at a complete result; but it often happens that the solution we obtain by the method of indeterminate coefficients involves no arbitrary constant. To supply this defect, we must introduce, instead of

the arbitrary constant, a value of y corresponding to a given value of x; that is, supposing these to be b and a, we must substitute in the given equation,

$$y = b + u, \quad x = a + t;$$

then determine u in a series all whose terms vanish when $t = 0$; and replace u and t by their values $y - b$ and $x - a$; in this way it is evident the arbitrary constant will be involved implicitly; for, from the complete integral

$$f(x, y, C) = 0,$$

C may be expressed in terms of a and b.

Ex. $\dfrac{dy}{dx} + y = gx^m,$

$$\text{this becomes } \frac{du}{dx} + b + u = g\,(a+t)^m;$$

$$\text{assume } u = t^\alpha\,(A + Bt^\beta + Ct^\gamma + \&\text{c.});$$

$$\begin{array}{llllll}
\therefore\ 0 = \alpha A t^{\alpha-1} & + (\alpha+\beta)\,Bt^{\alpha+\beta-1} & + (\alpha+\gamma)\,Ct^{\alpha+\gamma-1} & + \&\text{c.} \\
\quad + b & + \quad At^\alpha & + \quad Bt^{\alpha+\beta} & + \&\text{c.} \\
\quad - ga^m & - \quad mga^{m-1}t & - \dfrac{m(m-1)}{1.2}\,ga^{m-2}t^2 & - \&\text{c.}
\end{array}$$

$$\therefore\ \alpha = 1, \quad \beta = 1, \quad \gamma = 2, \ \&\text{c.}$$

$$A = ga^m - b, \quad 2B = gma^{m-1} - ga^m + b,$$

$$6C = gm\,(m-1)\,a^{m-2} - gma^{m-1} + ga^m - b, \ \&\text{c.}$$

87. The approximate solution of a differential equation may sometimes be obtained in the form of a continued fraction by assuming

$$y = \frac{Ax^\alpha}{1+}\,\frac{Bx^\beta}{1+}\,\frac{Cx^\gamma}{1+\&\text{c.}}.$$

First, suppose x to be very small, and for y substitute Ax^α in the given equation; then, retaining only the terms of lowest dimensions in x, A and α become known by equating the coefficients and exponents of those terms. Next, write $y = \dfrac{Ax^\alpha}{1+z}$

in the proposed equation, and in the result put $z = Bx^\beta$, and determine B, β, as before, by supposing x to be very small; then in the transformed equation in z, put $z = \dfrac{Bx^\beta}{1+t}$; and so on for the rest.

Ex. $$my + (1+x)\frac{dy}{dx} = 0.$$

Let $y = Ax^\alpha$; $\therefore\ (m+\alpha)Ax^\alpha + A\alpha x^{\alpha-1} = 0$,

$$\text{or } A\alpha x^{\alpha-1} = 0; \quad \therefore\ \alpha = 0,$$

and A remains undetermined. Next, put

$$y = \frac{A}{1+z}, \text{ and } z = Bx^\beta,$$

and we get successively $m(1+z) = (1+x)\dfrac{dz}{dx}$,

$$m + Bx^\beta(m-\beta) = \beta Bx^{\beta-1}, \quad \text{or } m = B\beta x^{\beta-1};$$

$$\therefore\ \beta = 1, \quad B = m;$$

similarly, putting $z = \dfrac{mx}{1+t}$, and $t = Cx^\gamma$,

we determine C and γ; and thus we obtain

$$y = \frac{A}{1+}\,\frac{mx}{1-}\,\frac{\frac{m-1}{1}\cdot\frac{x}{2}}{1+}\,\frac{\frac{m+1}{3}\cdot\frac{x}{2}}{1+}\,\frac{\frac{m-2}{3}\cdot\frac{x}{2}}{1-}\,\frac{\frac{m+2}{5}\cdot\frac{x}{2}}{1-}\,\frac{\frac{m-3}{5}\cdot\frac{x}{2}}{1+\&c.}.$$

Since the proposed equation when integrated gives

$$y = A(1+x)^{-m},$$

the above continued fraction is the development of $A(1+x)^{-m}$.

88. We may approximate to the integral of a differential equation by successive substitutions, in a manner similar to that employed by Newton for the solution of algebraical equations, as in the following instance.

Ex. $\dfrac{d^2y}{dx^2} + n^2y + \alpha y^2 + a = 0$, where α is a very small quantity.

We may assume

$$y = u + \alpha u_1 + \alpha^2 u_2 + \alpha^3 u_3 + \&c.$$

which gives

$$\frac{d^2u}{dx^2} + n^2u + a + \alpha\left(\frac{d^2u_1}{dx^2} + n^2u_1 + u^2\right)$$

$$+ \alpha^2\left(\frac{d^2u_2}{dx^2} + n^2u_2 + 2uu_1\right)$$

$$+ \alpha^3\left(\frac{d^2u_3}{dx^2} + n^2u_3 + 2uu_2\right) + \&c. = 0.$$

Hence, equating the coefficient of each power of α to zero, we get

$$\frac{d^2u}{dx^2} + n^2u + a = 0,$$

$$\frac{d^2u_1}{dx^2} + n^2u_1 + u^2 = 0,$$

$$\frac{d^2u_2}{dx^2} + n^2u_2 + 2uu_1 = 0, \ \&c. \ \ldots\ldots\ldots (3).$$

The first give $u = -\frac{a}{n^2} + c\cos nx + c'\sin nx$; and this value substituted in the second reduces it to the form

$$\frac{d^2u_1}{dx^2} + n^2u_1 = X_1,$$

the integral of which by Art. (67) may be shewn to be

$$u_1 = \cos nx\,(c_1 - \frac{1}{n}\int dx X_1 \sin nx) + \sin nx\,(c'_1 + \frac{1}{n}\int dx X_1 \cos nx).$$

Similarly, these values of u and u_1 substituted in (3) reduce it to the form

$$\frac{d^2u_2}{dx^2} + n^2u_2 = X_2,$$

which may be in like manner integrated; and in this way the coefficients of the powers of α may be deduced one from the other by a uniform process.

89. We have seen (Art. 63) that the solution of

$$\frac{d^2y}{dx^2} + n^2y = A\cos(mx + \alpha) + B\cos(nx + \beta),$$

is $$y = c_1 \cos nx + c_2 \sin nx + \frac{A}{n^2 - m^2} \cos (mx + \alpha) + \frac{B}{2n} x \sin (nx + \beta).$$

Hence, if from the proposed equation we had to determine y approximately, we could not neglect the term

$$A \cos (mx + \alpha)$$

even when A is exceedingly small, provided m and n are nearly equal to one another; because in the value of y this term is divided by $n^2 - m^2$ which is very small. With respect to the last term in the value of y, we remark, that it is not periodical, but may increase indefinitely, as x increases. The equations which present themselves for solution in physical Astronomy are usually of the above form; and upon the peculiarities just noticed depend some of the most interesting results in that subject.

The following example supplies an omission at the end of Art. 83. When m is an integer we have obtained (Art. 74, Ex. 4) the solution of the equation

$$\frac{d^2y}{dx^2} + n^2 y = \frac{m(m+1)}{x^2} y.$$

When m is a fraction, assume $y = x^{m+1} u$, then

$$\frac{d^2u}{dx^2} + \frac{2m+2}{x} \frac{du}{dx} + n^2 u = 0,$$

the complete integral of which is

$$u = \beta \int_{-n}^{n} dt \, (t^2 - n^2)^m \cos (xt + \alpha),$$

α and β being the arbitrary constants; for this gives

$$\frac{du}{dx} = -\beta \int_{-n}^{n} dt \, (t^2 - n^2)^m \sin (xt + \alpha) \,.\, t$$

$$= \frac{\beta x}{2m+2} \int_{-n}^{n} dt \, (t^2 - n^2)^{m+1} \cos (xt + \alpha),$$

$$\frac{d^2u}{dx^2} = -\beta \int_{-n}^{n} dt \, (t^2 - n^2)^m \cos (xt + \alpha) \,.\, t^2$$

$$= -\beta \int_{-n}^{n} dt \, (t^2 - n^2)^m (t^2 - n^2 + n^2) \cos (xt + \alpha);$$

$$\therefore \frac{d^2u}{dx^2} + n^2 u = -\beta \int_{-n}^{n} dt \, (t^2 - n^2)^{m+1} \cos (xt + \alpha) = -\frac{2m+2}{x} \frac{du}{dx}.$$

SECTION V.

ON DIFFERENTIAL EQUATIONS INVOLVING TWO OR MORE INDEPENDENT VARIABLES.

90. If u be a function of any number of independent variables x, y, z, t, &c., and if $u + \delta u$ be the value of u when these variables simultaneously become $x + \delta x$, $y + \delta y$, $z + \delta z$, &c.,

then $\delta u = \frac{du}{dx} \cdot \delta x + \frac{du}{dy} \cdot \delta y + \frac{du}{dz} \cdot \delta z + \ldots$

+ terms of two dimensions in δx, δy, &c.

If we now suppose all terms to be neglected of more than one dimension in δx, δy, &c., and denote the value of δu corresponding to that supposition by du, we have

$$du = \frac{du}{dx} \cdot \delta x + \frac{du}{dy} \cdot \delta y + \frac{du}{dz} \cdot \delta z + \ldots ;$$

according to which definition, it appears that du denotes that part of the increment of u as given by Taylor's Theorem, which involves only the first powers of the arbitrary increments of the variables on which it depends; hence, in conformity with this definition, δx, δy, δz, &c., must be represented by dx, dy, dx, &c.; for if $f(x) = x$, then

$$f(x + \delta x) = x + \delta x ;$$

consequently that part of the increment of $f(x)$ which involves the first power of δx is, in this case, the whole of it δx, therefore $dx = \delta x$; and so on for the others;

$$\therefore\ du = \frac{du}{dx} \cdot dx + \frac{du}{dy} \cdot dy + \frac{du}{dz} \cdot dz + \text{\&c.},$$

where dx, dy, dz, &c., are the arbitrary increments of the independent variables x, y, z, &c., and are entirely independent of

those variables; and du is that part of the corresponding increment of the dependent variable u which involves only simple dimensions of dx, dy, dz, &c.; and which approaches nearer and nearer to the value of the whole increment of u, the smaller dx, dy, dz, &c. are taken.

The quantities $\frac{du}{dx} . dx$, $\frac{du}{dy} . dy$, &c., are called the partial differentials of u with respect to x, y, &c. respectively; and du is called the total or complete differential of u, or the differential of u merely.

91. According to the above definition, the differential of du, or the second differential of u, will be, supposing it to involve only two independent variables,

$$d^2u = \frac{d}{dx}(du) . dx + \frac{d}{dy}(du) . dy$$

$$= \frac{d}{dx}\left(\frac{du}{dx} . dx + \frac{du}{dy} . dy\right) dx + \frac{d}{dy}\left(\frac{du}{dx} . dx + \frac{du}{dy} . dy\right) dy$$

$$= \frac{d^2u}{dx^2} . dx^2 + 2\frac{d}{dx}\frac{d}{dy} u . dxdy + \frac{d^2u}{dy^2} . dy^2,$$

because in this process dx and dy are independent of x and y; being, in fact, the arbitrary increments of those variables, which might have been denoted by h and k, did not a due regard to the precision and symmetry of the notation require it otherwise. And in general, if u be a function of any number of independent variables x, y, z, t, &c., the n^{th} differential of u will be given by the formula, (where the symbols of operation are separated from those of quantity,)

$$d^nu = (dx . \frac{d}{dx} + dy . \frac{d}{dy} + dz . \frac{d}{dz} + \&c.)^n u.$$

92. Hence we see that the differential of a function of several variables has no reference to one variable rather than to another; but in its formation, all the variables of which it is a function are supposed to undergo simultaneous unconnected alterations; and the value of the differential depends upon the

values of all those alterations. Whereas in forming a differential coefficient, one particular independent variable only is changed, the rest remaining unaffected; and a quantity is produced, wholly independent of the value of the alteration which that variable may have received.

It is evident that the rules for deriving differential coefficients suffice for finding the differentials of functions.

Having, for instance, formed all the partial differential coefficients of the first order, if we multiply each by the arbitrary increment or differential of the corresponding independent variable and take the sum, we shall obtain the first differential of the function; and similarly, the differentials of the second, and higher orders, may be formed by means of the formula at the end of the preceding Article.

Integration of Differential Functions of two or more Variables.

93. Total integration of a proposed differential is the finding a function whose differential is the quantity proposed. This operation is denoted by the symbol $\int$; the process, like the former of total differentiation, having reference to all and not to one in particular of the variables. When an expression presented for total integration can be reduced to the form $f(v)\,dv$, its integral $=\int dv f(v)$, and can be obtained immediately.

Ex. 1. If $u = x^2 + y^2$,

$$du = 2(xdx + ydy) \text{ is the differential of } u;$$

$$\text{also } \int (xdx + ydy) = \tfrac{1}{2}(x^2 + y^2),$$

since the latter quantity differentiated produces the quantity under the symbol of total integration.

Ex. 2. $$du = \frac{y\,(xdy - ydx)}{x^2\sqrt{x^2+y^2}} = \frac{\frac{y}{x}}{\sqrt{1+\frac{y^2}{x^2}}} \cdot \frac{xdy - ydx}{x^2}$$

$$= \frac{v}{\sqrt{1+v^2}}\,dv, \quad \text{putting } \frac{y}{x} = v;$$

$$\therefore \; u = \sqrt{1+v^2} + C = \frac{1}{x}\sqrt{x^2+y^2} + C.$$

94. To integrate $du = Pdx + Qdy$, u being a function of two independent variables x and y, and P and Q functions of x and y. This amounts to finding the integral of a differential function of two variables of the first order; and involves the same process as was used at Art. 7 for exact differential equations of the first order between two variables; since all such equations by being multiplied by an integrating factor are rendered differential functions.

Since $Pdx + Qdy$ must be identical with $\frac{du}{dx}dx + \frac{du}{dy}dy$, we have

$$\frac{du}{dx} = P, \quad \frac{du}{dy} = Q, \text{ with the condition } \frac{dP}{dy} = \frac{dQ}{dx};$$

$$\therefore \; u = \int dx\, P + f(y), \quad f(y) \text{ involving } y \text{ only};$$

$$\therefore \; Q = \frac{du}{dy} = \frac{d}{dy}(\int dx\, P) + \frac{d}{dy}f(y);$$

$$\therefore f(y) = \int dy\, \{Q - \frac{d}{dy}(\int dx\, P)\};$$

consequently u is known; and we may observe that in finding the value of $f(y)$ it is only necessary to integrate those terms of Q which involve y only. If we had begun by integrating with respect to y, the result would have been

$$u = \int dy\, Q + \int dx\, \{P - \frac{d}{dx}(\int dy\, Q)\};$$

and as before it is only necessary to integrate those terms of P which involve x only.

Ex. 1. $$du = \frac{y}{(x+y)^2}dx - \frac{x}{(x+y)^2}dy;$$

$$\text{here } \frac{du}{dx} = \frac{y}{(x+y)^2}, \quad \frac{du}{dy} = -\frac{x}{(x+y)^2},$$

$$\therefore \; u = -\frac{y}{x+y} + f(y);$$

and differentiating with respect to y

$$\frac{-1}{x+y}+\frac{y}{(x+y)^2}+\frac{d}{dy}f(y)=\frac{du}{dy}=-\frac{x}{(x+y)^2};$$

$$\therefore\ \frac{d}{dy}f(y)=0,\ \text{and}\ f(y)=C,$$

as we might have foreseen, since there is no term in Q that involves y only;

$$\therefore\ u=C-\frac{y}{x+y}.$$

Ex. 2. $$du=\frac{ydy+xdx-2xdy}{(x-y)^2}.$$

$$u=\log(x-y)-\frac{y}{x-y}+C.$$

95. To integrate $du=Pdx+Qdy+Rdz$, u being a function of three independent variables x, y, z.

Since $Pdx+Qdy+Rdz$ is identical with

$$\frac{du}{dx}.dx+\frac{du}{dy}.dy+\frac{du}{dz}.dz,$$

we have $$\frac{du}{dx}=P,\ \frac{du}{dy}=Q,\ \frac{du}{dz}=R;$$

together with the equations of condition

$$\frac{dP}{dy}=\frac{dQ}{dx},\ \frac{dQ}{dz}=\frac{dR}{dy},\ \frac{dR}{dx}=\frac{dP}{dz},$$

which are found by supposing each of the quantities z, x, y to be constant in succession (as we are evidently at liberty to do, since those variables are independent of one another), and then taking the corresponding equation of condition for du being the exact differential of a function of two independent variables. Hence

$$u=\int dxP+w,\ w\ \text{being a function of}\ y\ \text{and}\ z;$$

$$\text{also}\ \ Q=\frac{d}{dy}(\int dxP)+\frac{dw}{dy},$$

$$R=\frac{d}{dz}(\int dxP)+\frac{dw}{dz},$$

which two equations giving the values of the partial differential coefficients of w, its value may be found by the preceding Article; and so the value of u completely determined.

Ex. 1. $du = yzdx + xzdy + xydz$;

here $\frac{du}{dx} = yz$, $\frac{du}{dy} = xz$, $\frac{du}{dz} = xy$; $\therefore\ u = xyz + w$;

$$xz = xz + \frac{dw}{dy}, \quad \text{or} \quad \frac{dw}{dy} = 0;$$

$$xy = xy + \frac{dw}{dz}, \quad \text{or} \quad \frac{dw}{dz} = 0;$$

$$\therefore\ w = C, \text{ and } u = xyz + C.$$

Ex. 2. $du = \frac{y}{a-z}dx + \frac{x}{a-z}dy + \frac{xy}{(a-z)^2}dz$;

$$u = \frac{xy}{a-z} + C.$$

96. Differential equations involving more than two variables admit of division into two classes, Total and Partial. A total differential equation is one which expresses the relation between the differential of the dependent variable and the other variables and their differentials, and sometimes also the dependent variable itself; it is consequently equivalent to a system of equations in which each differential coefficient of the dependent variable is given explicitly. Thus z being a function of the independent variables x and y, the total differential equation

$$Pdx + Qdy + Rdz = 0,$$

in which P, Q, and R are functions of x, y and z, amounts to the same as the two equations

$$P + R\frac{dz}{dx} = 0, \quad Q + R\frac{dz}{dy} = 0.$$

A Partial differential equation, on the contrary, is a relation between all or certain of the partial differential coefficients of the dependent variable, and the variables: as in the instances

$$(x-a)\frac{dz}{dx}+(y-b)\frac{dz}{dy}=z-c,$$

$$x^2\frac{d^2z}{dx^2}+2\,xy\frac{d}{dx}\frac{d}{dy}z+y^2\frac{d^2z}{dy^2}=0,$$

z in both being a function of the independent variables x and y.

Total Differential Equations.

97. We may first consider the equation

$$Pdx+Qdy+Rdz=0,$$

one of the three variables x, y, z, being a function of the other two, which are independent.

Since the proposed equation may arise from combining the results of differentiating two separate equations, we have first to examine whether it can be satisfied by a single primitive relation between x, y and z. If x, y, z be co-ordinates of a point, the cases will be distinguished according as the proposed equation is the differential equation to a series of surfaces, or to a series of curves in space.

For example, the equation

$$(z-c)\,xdx+(z-c)\,ydy=\{x\,(x-a)+y\,(y-b)\}\,dz,$$

arises from combining the results of differentiating the equations

$$z=f(x^2+y^2),\quad \frac{y-b}{z-c}=\phi\left(\frac{x-a}{z-c}\right);$$

for these equations give respectively

$$y\frac{dz}{dx}-x\frac{dz}{dy}=0,\quad (x-a)\frac{dz}{dx}+(y-b)\frac{dz}{dy}=z-c;$$

from which if $\frac{dz}{dx}$ and $\frac{dz}{dy}$ be determined, and substituted in

$$dz=\frac{dz}{dx}\,.\,dx+\frac{dz}{dy}\,.\,dy,$$

we get the proposed equation; which, consequently, cannot in general be satisfied by a single relation between x, y, z. It is the

analytical expression of the conditions of the problem, to find a surface belonging at the same time to surfaces of revolution about the axis of z, and to conical surfaces.

98. To find the equation of condition for

$$Pdx + Qdy + Rdz = 0,$$

admitting a solution of the form $f(x, y, z) = C$.

If the proposed equation can be satisfied by a single relation between x, y and z, then

$$dz = -\frac{P}{R}dx - \frac{Q}{R}dy$$

is the differential of a function of two independent variables;

$$\therefore \frac{dz}{dx} = -\frac{P}{R}, \quad \frac{dz}{dy} = -\frac{Q}{R},$$

$$\text{with the condition } \frac{d}{dy}\left(\frac{P}{R}\right) = \frac{d}{dx}\left(\frac{Q}{R}\right);$$

or, since P, Q, R may contain z which is a function of x and y,

$$\frac{d}{dy}\left(\frac{P}{R}\right) + \frac{d}{dz}\left(\frac{P}{R}\right)\frac{dz}{dy} = \frac{d}{dx}\left(\frac{Q}{R}\right) + \frac{d}{dz}\left(\frac{Q}{R}\right)\frac{dz}{dx};$$

therefore, substituting for $\frac{dz}{dx}$, $\frac{dz}{dy}$ their values,

$$R\frac{d}{dy}\left(\frac{P}{R}\right) - Q\frac{d}{dz}\left(\frac{P}{R}\right) = R\frac{d}{dx}\left(\frac{Q}{R}\right) - P\frac{d}{dz}\left(\frac{Q}{R}\right);$$

and performing the differentiations and reducing we get

$$P\left(\frac{dR}{dy} - \frac{dQ}{dz}\right) + Q\left(\frac{dP}{dz} - \frac{dR}{dx}\right) + R\left(\frac{dQ}{dx} - \frac{dP}{dy}\right) = 0,$$

the equation of condition for

$$Pdx + Qdy + Rdz = 0$$

admitting a solution of the form $f(x, y, z) = C$.

99. If the above equation of condition be not satisfied, then the equation

$$Pdx + Qdy + Rdz = 0$$

cannot, by being multiplied by any factor, become susceptible of a solution of the form

$$f(x, y, z, C) = 0.$$

For suppose V to be a factor which renders

$$Pdx + Qdy + Rdz$$

the immediate differential of some function u of x, y, z considered as independent of one another;

$$\therefore \frac{d}{dy}(VP) = \frac{d}{dx}(VQ), \quad \frac{d}{dz}(VQ) = \frac{d}{dy}(VR),$$

$$\frac{d}{dx}(VR) = \frac{d}{dz}(VP);$$

$$\text{or} \quad \left.\begin{aligned} V\frac{dP}{dy} + P\frac{dV}{dy} &= V\frac{dQ}{dx} + Q\frac{dV}{dx} \\ V\frac{dQ}{dz} + Q\frac{dV}{dz} &= V\frac{dR}{dy} + R\frac{dV}{dy} \\ V\frac{dR}{dx} + R\frac{dV}{dx} &= V\frac{dP}{dz} + P\frac{dV}{dz} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ (1).$$

Hence, multiplying the first of these equations by R, the second by P, and the third by Q, and taking their sum, the factor V disappears, and we find

$$P\left(\frac{dR}{dy} - \frac{dQ}{dz}\right) + Q\left(\frac{dP}{dz} - \frac{dR}{dx}\right) + R\left(\frac{dQ}{dx} - \frac{dP}{dy}\right) = 0, \ \ldots\ldots\ (2)$$

the same equation of condition as in the preceding Article.

100. If this equation be satisfied, which will be the case only when the proposed admits a primitive of the form

$$f(x, y, z, C) = 0,$$

equations (1) afford a means of determining V. Then

$$du = VPdx + VQdy + VRdz,$$

whence u can be found by the method of Art. 95, and $u + C = 0$ is the required relation between x, y, and z.

Or, without determining V, we may integrate considering one of the variables constant, and add an arbitrary function of that variable; then differentiate the result with respect to that variable and compare it with the proposed equation, and so the function added will become known.

In the majority of cases which present themselves, the factor V is capable of being determined by inspection.

Ex. 1. $(ay - bz)\,dx + (cz - ax)\,dy + (bx - cy)\,dz = 0.$

Divide by $(ay - bz)\,bx - cy)$, and the resulting equation

$$\frac{dx}{bx - cy} + \frac{(cz - ax)\,dy}{(ay - bz)\,(bx - cy)} + \frac{dz}{ay - bz} = 0,$$

satisfies the conditions, considering x, y, z as independent,

$$\frac{dP}{dy} = \frac{dQ}{dx}, \quad \frac{dQ}{dz} = \frac{dR}{dy}, \quad \frac{dR}{dx} = \frac{dP}{dz};$$

and therefore the first member may be regarded as the differential of some function, u, of x, y, z considered as independent;

$$\therefore \frac{du}{dx} = \frac{1}{bx - cy}; \quad \text{and } u = \frac{1}{b} \log (bx - cy) + w;$$

$$\therefore \frac{1}{ay - bz} = \frac{dw}{dz},$$

$$\frac{cz - ax}{(ay - bz)\,(bx - cy)} = -\frac{c}{b} \cdot \frac{1}{bx - cy} + \frac{dw}{dy},$$

$$\text{or } \frac{dw}{dy} = \frac{a}{b} \cdot \frac{-1}{ay - bz}; \quad \text{hence } dw = -\frac{1}{b} \cdot \frac{ady - bdz}{ay - bz};$$

$$\therefore \; w = -\frac{1}{b} \log (ay - bz) + C;$$

$$\therefore \; u = \frac{1}{b} \log \left(\frac{bx - cy}{ay - bz}\right) + C = 0, \quad \text{or } \frac{bx - cy}{ay - bz} = C.$$

Ex. 2. $xdx + ydy + (x^2 + y^2)(1 - z^{-1})\,dz = 0, \; x^2 + y^2 = Cz^2e^{-2z}.$

When in an equation of this sort, the differentials enter above the first degree, it is not integrable unless it can be resolved

into rational factors of the form $Pdx + Qdy + Rdz$; for whatever be the integral, it must upon differentiation produce a result of that form.

101. If the equation

$$Pdx + Qdy + Rdz = 0$$

be susceptible of a primitive of the form $f(x, y, z, C) = 0$, and be homogeneous and of n dimensions with respect to x, y, z; then, putting $x = vz$, $y = wz$, and dividing by z^n, it becomes

$$S(vdz + zdv) + T(wdz + zdw) + Udz = 0,$$

S, T, U being functions of v and w;

$$\text{or} \quad -\frac{dz}{z} = \frac{Sdv + Tdw}{Sv + Tw + U};$$

hence the second member is an exact differential since the first is so, and it may be sometimes integrated by inspection, or by the method of Art. 94.

Ex. 1. $(y+z)\,dx + (x+z)\,dy + (x+y)\,dz = 0$;

$$\text{put } x = vz, \quad y = wz,$$

$$\therefore \frac{dz}{z} + \frac{(w+1)\,dv + (v+1)\,dw}{v(w+1) + w(v+1) + v + w} = 0,$$

$$\text{or } \frac{dz}{z} + \frac{1}{2}\frac{d(v + w + vw)}{v + w + vw} = 0;$$

$$\therefore \log z^2(v + w + vw) = \log C, \quad \text{or } xz + yz + xy = C.$$

Ex. 2. $(y^2 + yz + z^2)\,dx + (x^2 + xz + z^2)\,dy$

$$+ (x^2 + xy + y^2)\,dz = 0.$$

Make $x = vz$, $y = wz$,

$$\therefore \frac{dz}{z} + \frac{(w^2 + w + 1)\,dv + (v^2 + v + 1)\,dw}{v(w^2 + w + 1) + w(v^2 + v + 1) + v^2 + vw + w^2} = 0;$$

but of the latter fraction, the denominator

$$= vw(w + v + 1) + v(w + 1 + v) + w(v + 1 + w)$$

$$= (vw + v + w)(v + w + 1),$$

and the numerator

$$= (1 + v + w)\, d\,(v + w + vw) - (v + w + vw)\, d\,(v + w)\,;$$

$$\therefore\ \frac{dz}{z} + \frac{d\,(v + w + vw)}{v + w + vw} - \frac{d\,(1 + v + w)}{1 + v + w} = 0.$$

$$\therefore\ \log\left\{\frac{z\,(v + w + vw)}{1 + v + w}\right\} = \log C;$$

$$\therefore\ zx + zy + xy = C\,(z + x + y).$$

Ex. 3. $\quad 2\,(y + z)\,dx + (x + 3y + 2z)\,dy + (x + y)\,dz = 0.$

By the same substitution this becomes

$$\frac{3dz}{z} + 2\,\frac{dv + dw}{v + w} + \frac{dw}{w + 1} = 0\,; \quad \therefore\ (x + y)^2\,(y + z) = C.$$

Ex. 4. $\quad (x^2 - y^2 + z^2)\,dx - z^2dy + (y - x)\,(z^2 + xy + x^2)\dfrac{dz}{z} = 0\,;$

putting $x = vz,\ y = wz,$ this is reduced to

$$dw = (1 + v^2 - w^2)\,dv\,;$$

$$\therefore\ \frac{e^{-v^2}}{w - v} = \int dv e^{-v^2} + C, \quad \text{(Ex. 4. Art. 18)}$$

$$\text{or}\ \frac{z^2 e^{-\frac{x^2}{z^2}}}{y - x} = \int dx e^{-\frac{x^2}{z^2}} + Cz,$$

z being constant under the sign of integration.

102. We have seen that the equation

$$Pdx + Qdy + Rdz = 0,$$

when the equation of condition (Art. 98) is not satisfied, does not admit of being derived from a single primitive equation involving two independent variables. The integral in this case will be exhibited by a system of two equations; and the proposed equation cannot be regarded as the differential equation to a surface, but to a system of curves in space all endowed with some common property.

Ex. 1. $dz = aydx + bdy$.

Since the equation of condition in this case is not satisfied, x and y cannot be independent, and we may assume $y = f(x)$;

$$\therefore\ dz = af(x)\,dx + b\frac{d}{dx}f(x)\,dx,$$

$$\therefore\ z = a\int dx f(x) + bf(x),$$

which, with $y = f(x)$, constitutes the integral of the proposed equation.

In general, if V be a factor which makes $Pdx + Qdy$ an exact differential considering z as constant, and we find

$$\int (VPdx + VQdy) = w + \phi(z),$$

it is evident that

$$w + \phi(z) = 0, \text{ together with } \frac{dw}{dz} + \frac{d}{dz}\phi(z) - VR = 0,$$

satisfy the proposed equation, where $\phi(z)$ denotes any function of z.

Ex. 2. $zdx + xdy + ydz = 0$,

$$y + z\log x + \phi(z) = 0, \quad \log x + \frac{d}{dz}\phi(z) = \frac{y}{x}.$$

Ex. 3. $\{x(x-a) + y(y-b)\}\,dz = (z-c)(xdx + ydy)$,

$$x^2 + y^2 + 2\phi(z) = 0, \quad x(x-a) + y(y-b) + (z-c)\frac{d}{dz}\phi(z) = 0.$$

Partial Differential Equations. Equations of the first order.

103. In partial differential equations of two independent variables, the differential coefficients of the first order $\frac{dz}{dx}$, $\frac{dz}{dy}$, of the dependent variable z, are usually denoted by the symbols p and q; and $\frac{d^2z}{dx^2}$, $\frac{d^2z}{dxdy}$, $\frac{d^2z}{dy^2}$, the differential coefficients of the second order, by r, s, t, respectively. A partial differential equation is said to be of the n^{th} order, when it involves one or more of the partial differential coefficients of the dependent variable of the n^{th} order; but none of a superior

order. To be the general equation of the n^{th} order, it ought to contain the independent variables, and the dependent variable together with all its partial differential coefficients from the first order to the n^{th} order inclusive. To integrate a partial differential equation, is to find for the dependent variable an expression between the differential coefficients of which that relation exists which is indicated by the proposed equation; and under the most general form possible.

104. The complete integral of $f(x, y, z, p, q) = 0$, the general equation of the first order, will involve one arbitrary or general function.

For let $u = F\{x, y, z, \phi(v)\} = 0$, be an equation by virtue of which z is a function of the independent variables x and y, v being a known function of x, y, and z. Then p and q are given by the equations $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, each of which will involve $\frac{d}{dv}\phi(v)$ or $\phi'(v)$, and may involve $\phi(v)$; consequently, between the three equations $u = 0$, $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, it will be possible to eliminate $\phi(v)$ and $\phi'(v)$, and there will result a relation $f(x, y, z, p, q) = 0$, wholly independent of the form of the function $\phi(v)$; and it is evident that in general more than one function cannot thus be eliminated. Conversely, an equation of the form

$$f(x, y, z, p, q) = 0$$

being proposed, its complete integral to have all the generality possible must be of the form $F\{x, y, z, \phi(v)\} = 0$, where the form of $\phi(v)$ is arbitrary.

For example, let

$$u = z + mx + ny - \phi(v) = 0,$$

$$\text{where } v = (x-a)^2 + (y-b)^2 + (z-c)^2;$$

$$\text{then } \frac{du}{dx} = p + m - \phi'(v)\{2(x-a) + 2(z-c)p\} = 0,$$

$$\frac{du}{dy} = q + n - \phi'(v)\{2(y-b) + 2(z-c)q\} = 0.$$

Hence, transposing and dividing one result by the other to eliminate $\phi'(v)$, we find

$$\frac{p+m}{q+n}=\frac{(x-a)+(z-c)p}{(y-b)+(z-c)q},$$

or $p\{y-b-n(z-c)\}-q\{x-a-m(z-c)\}$

$$=n(x-a)-m(y-b),$$

the partial differential equation of which the complete integral is

$$z+mx+ny=\phi\{(x-a)^2+(y-b)^2+(z-c)^2\}.$$

105. To integrate an equation in which only one of the differential coefficients of the first order enters with x, y, and z.

Let the equation be

$$f(x, y, z, p)=0.$$

Integrate it, considering y as a constant, and in place of the arbitrary constant C, add a function of y of arbitrary form. The resulting solution, containing one arbitrary function, will have all the generality that can be attained.

Ex $$(x^2+y^2)\frac{dz}{dx}=z^2+y^2;$$

$$\therefore \frac{1}{z^2+y^2}\frac{dz}{dx}-\frac{1}{x^2+y^2}=0;$$

$$\therefore \tan^{-1}\frac{z}{y}-\tan^{-1}\frac{x}{y}=\tan^{-1}\phi(y),$$

$$\text{or } z-x=(y^2+xz)\phi(y),$$

$\phi(y)$ being arbitrary in form.

The equation $f(x, y, z, q)=0$ is similarly integrated, the correction in this case being an arbitrary function of x.

Ex. $$xy\frac{dz}{dy}+nz=0,$$

$$z^x y^n=\phi(x).$$

106. To integrate the equation of the first order,

$$Pp + Qq = R,$$

P, Q, and R being functions of x, y, and z.

Let the primitive be $F(x, y, z) = 0$; therefore, denoting $\frac{d}{dx} F(x, y, z)$ by $F'(x)$, and so on for the other differential coefficients, we get

$$F'(x) + F'(z) \cdot p = 0,$$

$$F'(y) + F'(z) \cdot q = 0;$$

$$\therefore \; PF'(x) + QF'(y) + RF'(z) = 0.$$

But $dF(x, y, z) = F'(x)\, dx + F'(y)\, dy + F'(z)\, dz = 0$,

$$\therefore \; PF'(x)\, dx + PF'(y)\, dy + PF'(z)\, dz = 0;$$

and $PF'(x)\, dx + QF'(y)\, dx + RF'(z)\, dx = 0$;

$$\therefore \; F'(y)\{Pdy - Qdx\} + F'(z)\{Pdz - Rdx\} = 0, \quad (1)$$

which is satisfied by

$$\left.\begin{array}{l} Pdy - Qdx = 0, \\ Pdz - Rdx = 0. \end{array}\right\} \quad (2).$$

Suppose that by integrating these equations either separately or conjointly, we obtain $M = a$, $N = b$, two relations between the three variables and the arbitrary constants a and b, which satisfy them. By means of the equations $M = a$, $N = b$, any two of the variables as y and z can be expressed in terms of a and b and the third variable x. The complete primitive then becomes

$$F(x, y, z) = \phi(x, a, b) = 0,$$

and the differential of $\phi(x, a, b)$ must by virtue of equations (2) be identically equal to zero, therefore $\phi(x, a, b)$ cannot contain x, and consequently

$$0 = \phi(a, b) = \phi(M, N), \text{ or } M = f(N).$$

The assumptions (2) satisfy equation (1) independently of the forms of $F'(y)$, $F'(x)$, that is, independently of the form of F; therefore the form of ϕ, and consequently of f, is arbitrary.

Hence $M=f(N)$ being a solution of the proposed equation and containing one arbitrary function, is the general primitive.

107. For the success of the method, it is in general necessary that of the three equations

$$Pdy - Qdx = 0, \quad Pdz - Rdx = 0, \quad Qdz - Rdy = 0,$$

one at least should contain those two variables only whose increments or differentials appear in it, or that by their combination such an equation should be produced. For by integrating it we shall obtain an equation by means of which one of the variables may be eliminated from either of the other auxiliary equations. They are called the reducing equations of

$$Pp + Qq = R;$$

and may be easily remembered under the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

108. By a similar process, partial differential equations of three or more independent variables can sometimes be integrated. If z be a function of x, y, t, and the equation be

$$Tn + Pp + Qq = R,$$

where $n = \frac{dz}{dt}$, we have

$$n = \frac{R - Pp - Qq}{T};$$

which, substituted in

$$dz = ndt + pdx + qdy,$$

$$\text{gives } Tdz - Rdt = p\,(Tdx - Pdt) + q\,(Tdy - Qdt).$$

And if the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{dt}{T}$, or

$$Tdz - Rdt = 0, \quad Tdx - Pdt = 0, \quad Tdy - Qdt = 0,$$

$$\text{give } L = a, \quad M = b, \quad N = c,$$

then the solution is

$$\phi\,(L, M, N) = 0, \quad \text{or } L = f(M, N).$$

109. To explain the geometrical meaning of the solution of a partial differential equation of the first order.

If we regard x, y, z as the co-ordinates of a point, and the proposed differential equation as the equation to a system of surfaces, $M=a$, $N=b$ are the equations to two surfaces which, being satisfied by the same values of x, y, z that satisfy the differential equation, conjointly represent a line situated on one of the surfaces represented by the primitive. By giving any alterations to a and b we obtain other lines in space; but in order that these lines may always lie on the surface in question, these alterations must not be independent but connected, according to some law expressed by the relation $a=f(b)$, or $M=f(N)$.

For example, the equation $px+qy=z$ is the differential equation to a conical surface, and

$$\frac{y}{z}=a, \quad \frac{x}{z}=b,$$

are the equations to any straight line through the vertex. If a and b undergo all possible variations consistently with the restriction imposed by the nature of the directrix, the assemblage of straight lines thus formed is the conical surface.

110. We shall now apply the preceding method, which is due to Lagrange, to some examples.

Ex. 1. $$px+qz+y=0.$$

This compared with $Pp+Qq=R$, gives in the place of

$$Qdz-Rdy=0, \quad Pdz-Rdx=0,$$

the reducing equations

$$zdz+ydy=0, \quad xdz+ydx=0.$$

From the former, which alone is immediately integrable, we get $z^2+y^2=a^2$; then by substitution in the latter

$$xdz+\sqrt{a^2-z^2}dx=0, \text{ or } \frac{dz}{\sqrt{a^2-z^2}}+\frac{dx}{x}=0;$$

$$\therefore \sin^{-1}\frac{z}{a}+\log x=\log b, \quad \text{or } xe^{\sin^{-1}\frac{z}{a}}=b;$$

but $a=\sqrt{z^2+y^2}$, therefore $\sin^{-1}\dfrac{z}{a}=\tan^{-1}\dfrac{z}{y}$; and $b=f(a)$,

$$\text{hence } xe^{\tan^{-1}\frac{z}{y}}=f(\sqrt{z^2+y^2}),$$

is the integral of the proposed equation.

Ex. 2. $p(x-a)+q(y-b)=z-c$.

Here the reducing equations are

$$(x-a)\,dz-(z-c)\,dx=0, \quad (y-b)\,dz-(z-c)\,dy=0;$$

$$\therefore \frac{x-a}{z-c}=\alpha, \quad \frac{y-b}{z-c}=\beta,$$

$$\therefore \frac{y-b}{z-c}=f\left(\frac{x-a}{z-c}\right).$$

Ex. 3. $$mp+nq=1;$$

$$\therefore mdz-dx=0, \quad ndz-dy=0;$$

$$\therefore x-mz=\alpha, \quad y-nz=\beta;$$

$$\therefore y-nz=f(x-mz).$$

Ex. 4. $$(x-mz)\,p+(y-nz)\,q=0;$$

$$\therefore (x-mz)\,dz=0; \quad \therefore z=a,$$

$$(x-mz)\,dy-(y-nz)\,dx=0,$$

$$\text{or } (x-ma)\,dy-(y-na)\,dx=0;$$

$$\frac{x-ma}{y-na}=b; \text{ but } a=f(b), \quad \therefore z=f\left(\frac{x-mz}{y-nz}\right).$$

5. $px+qy=nz, \quad z=x^nf\left(\dfrac{y}{x}\right)$.

6. $\dfrac{p}{y}+\dfrac{q}{x}=\dfrac{1}{z}, \quad z^2=xy+f\left(\dfrac{y}{x}\right)$.

7. $px^2+qy^2=z^2, \quad \dfrac{1}{z}=\dfrac{1}{x}+f\left(\dfrac{1}{y}-\dfrac{1}{x}\right)$.

8. $z-px-qy=n\sqrt{x^2+y^2+z^2}$,

$$x^{n-1}(z+\sqrt{x^2+y^2+z^2})=f\left(\frac{y}{x}\right).$$

9. $p(y+x)+q(y-x)=z,$

$$\tan^{-1}\frac{x}{y}-\log(x^2+y^2)^{\frac{1}{2}}=f\left(\frac{z}{\sqrt{x^2+y^2}}\right).$$

10. $x\dfrac{dz}{dx}+y\dfrac{dz}{dy}+t\dfrac{dz}{dt}+u\dfrac{dz}{du}=nz,\ z=t^nf\left(\dfrac{x}{t},\ \dfrac{y}{t},\ \dfrac{u}{t}\right).$

111. The following examples require artifices in the combination of the reducing equations.

Ex. 1. $(2az+cx)\,p+(2bz-cy)\,q=(ay+bx)\,c.$

The reducing equations are

$$(ay+bx)\,cdx=(2az+cx)\,dz,$$

$$(ay+bx)\,cdy=(2bz-cy)\,dz.$$

Multiply the first by y, and the second by x, and add them together, then

$$cydx+cxdy=2zdz,\ \text{ or } z^2-cxy=\alpha.$$

Multiply the first by $-b$, and the second by a, and add them together, then

$$-bdx+ady=-dz,\ \text{ or } z+ay-bx=\beta;$$

$$\therefore\ z+ay-bx=\phi\,(z^2-cxy).$$

Ex. 2. $\{y-b-n(z-c)\}\,p-\{x-a-m(z-c)\}\,q$

$$=n(x-a)-m(y-b),$$

the equation to surfaces of revolution. Here the reducing equations are

$$\{n(x-a)-m(y-b)\}\,dx-\{y-b-n(z-c)\}\,dz=0,\quad (1).$$

$$\{y-b-n(z-c)\}\,dy+\{x-a-m(z-c)\}\,dx=0,\quad (2).$$

$$\{x-a-m(z-c)\}\,dz+\{n(x-a)-m(y-b)\}\,dy=0,\quad (3).$$

Multiply (1) by $x-a$, and (3) by $y-b$, and add, then

$$\{n(x-a)-m(y-b)\}\,\{(x-a)\,dx+(y-b)\,dy+(z-c)\,dz\}=0;$$

$$\therefore\ (x-a)\,dx+(y-b)\,dy+(z-c)\,dz=0;$$

$$\text{or } (x-a)^2+(y-b)^2+(z-c)^2=\alpha.$$

Multiply (1) by m, and (3) by n, and add, then

$$\{n(x-a)-m(y-b)\}(mdx+ndy+dz)=0;$$

$$\therefore\ mdx+ndy+dz=0,$$

$$\text{or } mx+ny+z=\beta=f(a);$$

$$\therefore\ z+mx+ny=f\{(x-a)^2+(y-b)^2+(z-c)^2\}.$$

Ex. 3. $(y+z)p+(x+z)q=x+y.$

The reducing equations are

$$(y+z)\,dz=(x+y)\,dx,$$

$$(x+z)\,dz=(x+y)\,dy;$$

$$\therefore\ (y-x)\,dz=-(x+y)(dy-dx)$$

$$(x+y)(dx+dy+dz)=2(x+y+z)\,dz;$$

$$\therefore\ \frac{dx+dy+dz}{x+y+z}=\frac{2dz}{x+y}=-\frac{2(dy-dx)}{y-x};$$

$$\therefore\ (x+y+z)(y-x)^2=\alpha.$$

Similarly, $$(x+y+z)(y-z)^2=\beta;$$

$$\therefore\ F\{(x+y+z)\,.\,(y-x)^2,\ \ (x+y+z)\,.\,(z-y)^2\}=0.$$

Ex. 4. $$(x+y+z)\frac{dz}{dt}+(t+y+z)\frac{dz}{dx}$$

$$+(t+x+z)\frac{dz}{dy}=t+x+y.$$

The reducing equations are

$$(x+y+z)\,dz=(t+x+y)\,dt,$$

$$(x+y+z)\,dx=(t+y+z)\,dt,$$

$$(x+y+z)\,dy=(t+x+z)\,dt;$$

$$\therefore\ (x+y+z)(dz-dt)=(t-z)\,dt,$$

$$(x+y+z)(dx-dt)=(t-x)\,dt,$$

$$(x+y+z)(dy-dt)=(t-y)\,dt,$$

$$(x+y+z)(dx+dy+dz+dt)=3(t+x+y+z)\,dt;$$

$$\text{hence } \frac{dz-dt}{t-z}=\frac{dt}{x+y+z}=\frac{dx+dy+dz+dt}{3(x+y+z+t)};$$

$$\therefore\ (x+y+z+t).(z-t)^3=\alpha;\ \text{similarly}$$

$$(x+y+z+t).(x-t)^3=\beta,\quad (x+y+z+t).(y-t)^3=\gamma;$$

$$\therefore\ (x+y+z+t).(z-t)^3=$$

$$F\{(x+y+z+t).(x-t)^3,\ (x+y+z+t).(y-t)^3\}.$$

Method of solving linear Partial Differential Equations by separation of symbols.

112. As the symbols of operation here employed $\frac{d}{dx}$, $\frac{d}{dy}$ are independent of one another, and as they are subject in their combinations to the same laws as algebraical quantities, we may by separating the symbols of operation from those of quantity readily effect the solution of many partial differential equations by the same processes as were employed for differential equations between two variables of the like class. In particular the whole class of Linear Partial Differential Equations bears a strict analogy to equations of the same sort between two variables, both in the mode of treatment which they admit, and in the form of the solution obtained.

Great use will be here made of the symbolical expressions for Taylor's Theorem, applied to a function of one or more variables: viz.

$$f(x+h)=e^{h\frac{d}{dx}}f(x),\quad f(x+h,\,y)=e^{h\frac{d}{dx}}f(x,\,y),$$

$$f(x+h,\,y+k)=e^{h\frac{d}{dx}+k\frac{d}{dy}}f(x,\,y);$$

and this use of the symbols $e^{h\frac{d}{dx}}$, $e^{k\frac{d}{dy}}$, must be particularly attended to, as having an important bearing on the interpretation of the results obtained. For since, when only two independent variables x and y are involved, after every integration relative to x an arbitrary function of y, $\phi(y)$, must be added instead of an arbitrary constant, if an operating factor such as $\left(\frac{d}{dx}-a\frac{d}{dy}\right)^{-1}$

be applied to a subject $f(x, y)$, we shall have by Art. 53, (taking notice that in operating relative to x, we must treat y and the symbol $\dfrac{d}{dy}$ like simple constants; and the same holds relative to x and $\dfrac{d}{dx}$ when integrating or differentiating relative to y),

$$\left(\frac{d}{dx} - a\frac{d}{dy}\right)^{-1} f(x, y) = e^{ax\frac{d}{dy}}\{\int dx\, e^{-ax\frac{d}{dy}} f(x, y) + \phi(y)\}$$

$$= e^{ax\frac{d}{dy}} \int dx\, f(x, y - ax) + \phi(y + ax),$$

where we see that an arbitrary function of a binomial, viz. $\phi(y + ax)$ here takes the place of the term Ce^{ax} consisting of an arbitrary constant multiplied by the exponential quantity e^{ax}, which is always met with in the solutions of linear equations between two variables. In the following examples we shall often for the sake of convenience write d, d', d'' instead of $\dfrac{d}{dx}$, $\dfrac{d}{dy}$, $\dfrac{d}{dt}$, suppressing the differentials; and arbitrary functions of x and y will sometimes be denoted by c_x, c_y.

113. We shall begin with the linear equation of the first order having constant coefficients,

$$a\frac{dz}{dx} + b\frac{dz}{dy} + cz = f(x, y),$$

or $\left(a\dfrac{d}{dx} + bd' + c\right) z = f(x, y)$;

$$\therefore\ az = \left(\frac{d}{dx} + \frac{c + bd'}{a}\right)^{-1} f(x, y) = e^{-\frac{x}{a}(c+bd')}\{\int dx\, e^{\frac{x}{a}(c+bd')} f(x, y) + \phi(y)\},$$

or $$e^{\frac{cx}{a}}\, az = e^{-\frac{bx}{a}d'} \int dx\, e^{\frac{cx}{a}} f\left(x, y + \frac{bx}{a}\right) + \phi\left(y - \frac{bx}{a}\right).$$

Suppose $f(x, y)$ to be a rational integral algebraic function of n dimensions;

$$\text{then } z = \frac{1}{c}\{1 + \frac{1}{c}(ad + bd')\}^{-1} f(x, y)$$

$$= \frac{1}{c}\{1 - \frac{1}{c}(ad + bd') + \frac{1}{c^2}(ad + bd')^2 - \&\text{c.} \pm \frac{1}{c^n}(ad + bd')^n\} f(x, y)$$

$$+ e^{-\frac{cx}{a}} \phi\left(y - \frac{bx}{a}\right);$$

where we stop with the n^{th} term of the development of the operating function, because any power of d or d' higher than the n^{th}, applied to $f(x, y)$, produces zero. Hence if $f(x, y) = g$, a constant,

$$z = \frac{g}{c} + e^{-\frac{cx}{a}} \phi\left(y - \frac{bx}{a}\right);$$

unless $c = 0$, when $z = \dfrac{gx}{a} + \phi\left(y - \dfrac{bx}{a}\right)$; and if $f(x, y) = gxy$,

$$z = g\,\frac{xy}{c} - g\,\frac{ay + bx}{c^2} + \frac{2gab}{c^3} + e^{-\frac{cx}{a}} \phi\left(y - \frac{bx}{a}\right).$$

Again, supposing $f(x, y)$ to be equal to ge^{mx+ny}, or to $g\cos(mx + ny)$, the values of z will be found to be, respectively,

$$z = \frac{ge^{mx+ny}}{am + bn + c} + e^{-\frac{cx}{a}} \phi\left(y - \frac{bx}{a}\right);$$

$$z = g \cdot \frac{c\cos(mx + ny) + (ma + nb)\sin(mx + ny)}{c^2 + (ma + nb)^2} + e^{-\frac{cx}{a}} \phi(ay - bx).$$

114. If the proposed equation be

$$a\frac{dz}{dx} + b\frac{dz}{dy} + cz^2 = zf(x, y),$$

then assuming $z = e^v$, or $v = \log z$, where v denotes a new function of x and y, the equation is reduced to the same form as the preceding, viz.:

$$a\frac{dv}{dx} + b\frac{dv}{dy} + cv = f(x, y),$$

hence the solution is

$$ae^{\frac{cx}{a}}\log z = e^{-\frac{bx}{a}c'}\int dx e^{\frac{cx}{a}} f\left(x,\, y+\frac{bx}{a}\right) + \phi\left(y-\frac{bx}{a}\right).$$

Thus let $f(x, y) = \dfrac{ay+\beta x}{(ay-bx)\,y} = \dfrac{1}{b}\cdot\dfrac{ab+\beta a}{ay-bx} - \dfrac{\beta}{by}$;

$$\therefore f\left(x,\, y+\frac{bx}{a}\right) = \frac{ab+\beta a}{ab}\cdot\frac{1}{y} - \frac{a\beta}{b}\,\frac{1}{ay+bx};$$

$$\therefore\ a\log z = \frac{ab+\beta a}{b}\cdot\frac{x}{ay-bx} - \frac{a\beta}{b^2}\log(ay) + \phi\left(y-\frac{bx}{a}\right)$$

is the solution (where $c=0$) of

$$a\frac{dz}{dx} + b\frac{dz}{dy} = \frac{ay+\beta x}{ay-bx}\cdot\frac{z}{y}.$$

A particular case of this is

$$\frac{dz}{dx} - m\frac{dz}{dy} = \frac{az}{y} + \frac{bz}{mx+y},$$

$$\text{for which } z = y^{-\frac{a}{m}}\,.\,e^{\frac{bx}{y+mx}}\,.\,\phi(y+mx).$$

Again, let $f(x, y) = \dfrac{\alpha}{x} + \dfrac{\beta}{y} + \dfrac{\gamma}{\sqrt{xy}}$,

then $\int dx f\left(x,\, y+\dfrac{bx}{a}\right) = \alpha\log x + \dfrac{a\beta}{b}\log\left(y+\dfrac{bx}{a}\right)$

$$+\, 2\gamma\sqrt{\frac{a}{b}}\log\left(\sqrt{x} + \sqrt{\frac{ay}{b}+x}\right);$$

$$\therefore\ a\log z = \alpha\log x + \frac{a\beta}{b}\log y + 2\gamma\sqrt{\frac{a}{b}}\log\left(\sqrt{x}+\sqrt{\frac{ay}{b}}\right) + \phi\left(y-\frac{bx}{a}\right)$$

is the solution (where $c=0$) of

$$a\frac{dz}{dx} + b\frac{dz}{dy} = z\left(\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{\sqrt{xy}}\right).$$

A particular case of this is

$$\frac{dz}{dx}+m\frac{dz}{dy}=z\left(\frac{\alpha}{x}+\frac{\beta}{y}+\frac{\gamma}{\sqrt{xy}}\right),$$

for which $z=x^{\alpha}y^{\frac{\beta}{m}}\left(\sqrt{x}+\sqrt{\frac{y}{m}}\right)^{\frac{2\gamma}{\sqrt{m}}}.\phi\,(y-mx).$

115. Next let an equation involving three independent variables be proposed,

$$\frac{dz}{dx}+a\frac{dz}{dy}+b\frac{dz}{dt}=x^2yt,$$

$$\text{or}\quad (d+ad'+bd'')\,z=x^2yt\,;$$

$$\therefore\ z=\frac{1}{d}\left(1+\frac{ad'+bd''}{d}\right)^{-1}x^2yt$$

$$=\left(\frac{1}{d}-\frac{ad'+bd''}{d^2}+\frac{2abd'd''}{d^3}\right)x^2yt,$$

these being the only terms necessary to be preserved in the development of the operating function, since $d'^2yt=0$; therefore, effecting the operations indicated,

$$z=\frac{1}{3}x^3yt-\frac{1}{12}(at+by)\,x^4+\frac{1}{30}abx^5+\phi\,(y-ax,\ t-bx)\,;$$

the complementary function being

$$e^{-x(ad'+bd'')}\phi\,(y,t)=\phi\,(y-ax,\ t-bx).$$

116. We come next to the case of the complete linear equation of the second order with constant coefficients

$$(ad^2+bdd'+cd'^2+\alpha d'+\beta d+\gamma)\,z=f(x,y).$$

This may be integrated whenever the operating function can be resolved into two factors rational with respect to d and d'; the condition for which may be shewn, by putting the operating function equal to zero and solving it relative to d, to be

$$(b^2-4ac)\,\gamma=b\alpha\beta-a\alpha^2-c\beta^2.$$

Thus, suppose $b=1$, $a=c=0$, then $\gamma=\alpha\beta$, and

$$(dd'+\alpha d'+\beta d+\alpha\beta)\,z=f(x,y),$$

$$\text{or } \left(\frac{d}{dx}+\alpha\right)\left(\frac{d}{dy}+\beta\right)z=f(x,y);$$

$$\therefore \left(\frac{d}{dy}+\beta\right)z=\left(\frac{d}{dx}+\alpha\right)^{-1}f(x,y)=e^{-\alpha x}\int dx e^{\alpha x}f(x,y)+e^{-\alpha x}C_y,$$

$$\therefore z=e^{-\alpha x-\beta y}\int dy\,e^{\beta y}\{\int dx e^{\alpha x}f(x,y)\}+e^{-\alpha x-\beta y}\int dy\,e^{\beta y}C_y+e^{-\beta y}C_x'.$$

Hence, changing the arbitrary functions, we find

$$z=e^{-\alpha x-\beta y}\int dy e^{\beta y}\{\int dx e^{\alpha x}f(x,y)\}+e^{-\alpha x}c_y+e^{-\beta y}c'_x,$$

for the solution of

$$\frac{d^2z}{dxdy}+\alpha\frac{dz}{dy}+\beta\frac{dz}{dx}+\alpha\beta z=f(x,y).$$

Hence if $f(x,y)=e^{mx+ny}$, since in this case $\frac{d}{dx}$ and $\frac{d}{dy}$ are equivalent respectively to factors m and n,

$$z=\frac{e^{mx+ny}}{(m+\alpha)(n+\beta)}+e^{-\alpha x}c_y+e^{-\beta y}c'_x.$$

Again, let $f(x,y)=e^{mx}\cos ny$, then

$$\int dx e^{\alpha x}f(x,y)=\frac{e^{(m+\alpha)x}\cos ny}{m+\alpha};$$

$$\therefore z=\frac{e^{mx-\beta y}}{m+\alpha}\int dy\,e^{\beta y}\cos ny$$

$$=\frac{e^{mx}}{m+\alpha}\,\frac{n\sin ny+\beta\cos ny}{n^2+\beta^2}+e^{-\alpha x}c_y+e^{-\beta y}c'_x.$$

Similarly, if we suppose $a=1$, $b=\gamma=0$, then $c=-\frac{\alpha^2}{\beta^2}$, and the equation is

$$(d^2-\frac{\alpha^2}{\beta^2}d'^2+\alpha d'+\beta d)\,z=f(x,y),$$

$$\text{or } (d+\frac{\alpha}{\beta}d')\,(d-\frac{\alpha}{\beta}d'+\beta)\,z=f(x,y).$$

If we assume $f(x,y)=0$, this gives

$$z=e^{-\beta x}e^{\frac{\alpha x}{\beta}d}\phi(y)+e^{-\frac{\alpha x}{\beta}d}\psi(y)=e^{-\beta x}\phi\left(y+\frac{\alpha x}{\beta}\right)+\psi\left(y-\frac{\alpha x}{\beta}\right)$$

for the solution of

$$\frac{d^2z}{dx^2}-\frac{\alpha^2}{\beta^2}\frac{d^2z}{dy^2}+\beta\frac{dz}{dx}+\alpha\frac{dz}{dy}=0.$$

If $b^2 = 4ac$, we must have

$$b\alpha\beta = a\alpha^2 + c\beta^2, \quad \text{or } b\beta = 2a\alpha;$$

then the proposed equation becomes

$$\left\{\left(\sqrt{a}\,d + \sqrt{c}\,d' + \frac{\beta}{2\sqrt{a}}\right)^2 - \left(\frac{\beta^2}{4a} - \gamma\right)\right\} z = f(x, y);$$

and the operating function can be separated into two real factors, if $\beta^2 > 4a\gamma$.

117. Next, let the equation be of the n^{th} order

$$\left(\frac{d}{dx}\right)^n z + p_1\left(\frac{d}{dx}\right)^{n-1}\frac{d}{dy}z + p_2\left(\frac{d}{dx}\right)^{n-2}\left(\frac{d}{dy}\right)^2 z + \ldots + p_n\left(\frac{d}{dy}\right)^n z = V,$$

the index of differentiation being the same in every term; and V denoting a function of x and y; then the equation may be written

$$\left\{\left(\frac{d}{d'}\right)^n + p_1\left(\frac{d}{d'}\right)^{n-1} + p_2\left(\frac{d}{d'}\right)^{n-2} + \ldots + p_n\right\} d'^n z = V.$$

Let a be a root that occurs singly, and b a root that occurs m times, in the auxiliary equation

$$v^n + p_1 v^{n-1} + p_2 v^{n-2} + \ldots + p_n = 0;$$

then by resolution into partial fractions we shall have

$$\frac{1}{v^n + p_1 v^{n-1} + \&c.} = \frac{A}{v-a} + \frac{B_m}{(v-b)^m} + \frac{B_{m-1}}{(v-b)^{m-1}} + \ldots + \frac{B_1}{v-b} + \&c.$$

Hence substituting $\dfrac{d}{d'}$ for v, we get

$$z = \left\{\frac{Ad'}{d-ad'} + \frac{B_m d'^m}{(d-bd')^m} + \frac{B_{m-1}d'^{m-1}}{(d-bd')^{m-1}} + \ldots + \frac{B_1 d'}{d-bd'} + \&c.\right\} d'^{-n} V;$$

$$\therefore\ z = Ae^{axd'}\int dx\, e^{-axd'} d'^{-n+1} V + B_m e^{bxd'} \left(\int dx\right)^m e^{-bxd'} d'^{-n+m} V + \&c.$$

Instead of reserving the complementary functions under the sign of integration, we may obtain them explicitly by supposing $V=0$; then since

$$\int dx\, 0 = c_v, \quad \left(\int dx\right)^m 0 = c_v^{\,0} + xc_v{}' + x^2 c_v{}'' + \ldots + x^{m-1}c_v{}^{(m-1)},$$

the part of the complementary function due to the root a and the m roots b is, changing the arbitrary functions,

$$Ac_{y+ax} + B_m\,(c^0_{y+bx} + xc'_{y+bx} + x^2c''_{y+bx} + \dots + x^{m-1}c^{(m-1)}_{y+bx})\ ;$$

and similar terms will be introduced by the other roots of the auxiliary equation.

Ex. 1. $\dfrac{d^2z}{dx^2} - n^2\dfrac{d^2z}{dy^2} = cxy$;

$$\therefore\ z = \frac{cxy}{d^2 - n^2d'^2} = d^{-2}\left\{1 + n^2\left(\frac{d'}{d}\right)^2 + \&\text{c.}\right\} cxy = d^{-2}cxy,$$

all the other terms being neglected, because when the operations are performed they vanish of themselves;

$$\therefore\ z = \frac{c}{6}x^3y + c_{y+nx} + c'_{y-nx}.$$

Ex. 2. $\left(\dfrac{d}{dx}\right)^3 z - 4\left(\dfrac{d}{dx}\right)^2\dfrac{d}{dy}z + 5\dfrac{d}{dx}\left(\dfrac{d}{dy}\right)^2 z - 2\left(\dfrac{d}{dy}\right)^3 z = e^{mx+ny}$,

$$\text{or}\ \ (d - 2d')(d - d')^2 z = e^{mx+ny};$$

$$\therefore\ z = \frac{e^{mx+ny}}{(m - 2n)(m - n)^2} + c_{y+2x} + c^0_{y+x} + xc'_{y+x},$$

since d and d' operating upon e^{mx+ny} are equivalent respectively to the factors m and n.

If the second member of the equation be $x^2 + y^2$, then

$$z = \frac{x^2 + y^2}{d^3 - 4d^2d' + 5dd'^2 - 2d'^3} = d^{-3}\left(1 + 4\frac{d'}{d} + 11\frac{d'^2}{d^2}\right)(x^2 + y^2),$$

these being the only terms in the development of the operating function which it is necessary to retain;

$$\therefore\ z = \frac{x^3y^2}{6} + \frac{x^4y}{3} + \frac{x^5}{5} + c_{y+2x} + c^0_{y+x} + xc'_{y+x}.$$

Ex. 3. $\dfrac{d^2z}{dx^2} + (a + b)\dfrac{d^2z}{dxdy} + ab\dfrac{d^2z}{dy^2} = f(x, y)$,

$$\text{or}\ \ (d + ad')(d + bd')\,z = f(x, y)\,;$$

$$\therefore\ z = \frac{d'^{-1}}{a-b}\left(\frac{1}{d+bd'} - \frac{1}{d+ad'}\right) f(x, y),$$

$$= \frac{d'^{-1}}{a-b}\left\{e^{-bxd'}\int dx f(x, y+bx) - e^{-axd'}\int dx f(x, y+ax)\right\}.$$

Let $f(x, y) = 0$, then $(d+ad')\ (d+bd')\ z = 0$ gives

$$z = e^{-axd'}\phi(y) + e^{-bxd'}\psi(y) = \phi(y-ax) + \psi(y-bx),$$

which shews the form of the complementary functions, whatever be the second member of the equation. If $f(x, y)$ be a rational integral algebraic function of x and y, then

$$z = [d^{-2} - d^{-4}\{(a+b)dd' + abd'^2\} + \&c.]\, f(x, y) + \phi(y-ax) + \psi(y-bx).$$

Thus let $\qquad f(x, y) = c + m(x+y) + nxy,$

then $z = \{d^{-2} - (a+b)\, d^{-3} d'\}\ \{c + m(x+y) + nxy\}$

$$= \frac{1}{2}(c+my)\,x^2 + \frac{1}{6}(m+ny)\,x^3 - \frac{1}{24}(a+b)\,(4m+nx)\,x^3$$

$$+ \phi(y-ax) + \psi(y-bx).$$

Change of the Independent Variables.

118. The method by changing the independent variables is applicable to partial differential equations, and will generally lead to the solution of any equation of a form analogous to one of those for which the method succeeds in the case of differential equations between two variables. The change of the independent variables that has the most extensive application is to assume $x = e^t$, $y = e^v$, as we proceed to shew in the following instances, making use of the formulæ of Art. 70.

Ex. 1. $$ax\frac{dz}{dx} + by\frac{dz}{dy} + cz = V;$$

this becomes

$$a\frac{dz}{dt} + b\frac{dz}{dv} + cz = U,$$

the integral of which has already been found (Art. 113).

Let $V = x^m y^n$, then $U = e^{mt+nv}$;

$$\therefore\ z = \frac{e^{mt+nv}}{am+bn+c} + e^{-\frac{ct}{a}}\phi\left(v - \frac{bt}{a}\right),$$

or, restoring the values of t and v, and changing the arbitrary function,

$$z = \frac{x^m y^n}{am + bn + c} + x^{-\frac{c}{a}} \phi\left(\frac{y^a}{x^b}\right).$$

Ex. 2. $$ax^2 \frac{d^2z}{dx^2} + (a+b)\, xy \frac{d^2z}{dxdy} + by^2 \frac{d^2z}{dy^2} + cx \frac{dz}{dx}$$

$$+ cy \frac{dz}{dy} - cz = 0.$$

This, by the same substitutions as in the foregoing example, becomes

$$ad\,(d-1)\,z + (a+b)\, dd'z + bd'\,(d'-1)\,z + c\,(d+d'-1)\,z = 0,$$

$$\text{or} \;\; (ad + bd' + c)\,(d + d' - 1)\,z = 0;$$

$$\therefore \;\; az = e^{-\frac{ct}{a}} e^{-\frac{bt}{a}d'} \phi(v) + e^{t(1-d')} \psi(v)$$

$$= e^{-\frac{ct}{a}} \phi\left(v - \frac{bt}{a}\right) + e^t \psi(v - t);$$

hence, restoring the values of v and t,

$$z = x^{-\frac{c}{a}} \phi\,(yx^{-\frac{b}{a}}) + x\psi\left(\frac{y}{x}\right).$$

Ex. 3. $$ax^3 \frac{d^3z}{dx^3} + (2a+b)\, x^2y \frac{d^3z}{dx^2dy}$$

$$+ (a+2b)\, xy^2 \frac{d^3z}{dxdy^2} + by^3 \frac{d^3z}{dy^3} = 0;$$

$$\therefore \;\; \{ad\,(d-1)\,(d-2) + (2a+b)\, d\,(d-1)\, d'$$

$$+ (a+2b)\, dd'(d'-1) + bd'(d'-1)\,(d'-2)\}\, z = 0,$$

$$\text{or} \;\; (ad + bd')\,(d + d' - 1)\,(d + d' - 2)\,z = 0;$$

$$\therefore \;\; z = e^{-\frac{bt}{a}d'} \phi\,(v) + e^{-t\,(d'-1)} \psi\,(v) + e^{-t\,(d'-2)} \chi\,(v)$$

$$= \phi\left(v - \frac{bt}{a}\right) + e^t \psi\,(v - t) + e^{2t} \chi\,(v - t)$$

$$= \phi\,(yx^{-\frac{b}{a}}) + x\,\psi\left(\frac{y}{x}\right) + x^2 \chi\left(\frac{y}{x}\right).$$

Ex. 4. $x^n \dfrac{d^n z}{dx^n} + nx^{n-1} y \dfrac{d^n z}{dx^{n-1} dy} + \dfrac{n(n-1)}{1 \,.\, 2} x^{n-2} y^2 \dfrac{d^n z}{dx^{n-2} dy^2} + \text{\&c.}$

$+ y^n \dfrac{d^n z}{dy^n} = 0.$ Here we have (Art. 70)

$$\{d(d-1) \dots (d-n+1) + nd(d-1) \dots (d-n+2) d'$$

$$+ \frac{n(n-1)}{1 \,.\, 2} d(d-1) \dots (d-n+3) d'(d'-1) + \text{\&c.}\} z = 0,$$

$$\text{or } (d+d')(d+d'-1) \dots (d+d'-n+1) z = 0,$$

as is easily seen by equating the coefficients of u^n in the product of the expansions of $(1+u)^d$, $(1+u)^{d'}$, and in the equivalent expansion of $(1+u)^{d+d'}$. Hence

$$z = e^{-td'} f_0(v) + e^{-t(d'-1)} f_1(v) + \dots + e^{-t(d'-n+1)} f_{n-1}(v)$$

$$= f_0(v-t) + e^t f_1(v-t) + \dots + e^{(n-1)t} f_{n-1}(v-t)$$

$$= f_0\left(\frac{y}{x}\right) + x f_1\left(\frac{y}{x}\right) + \dots + x^{n-1} f_{n-1}\left(\frac{y}{x}\right).$$

Ex. 5. $\left(x\dfrac{d}{dx} + y\dfrac{d}{dy}\right)^n z + p_1 \left(x\dfrac{d}{dx} + y\dfrac{d}{dy}\right)^{n-1} z + \dots + p_n z = 0.$

Since $\left(x\dfrac{d}{dx} + y\dfrac{d}{dy}\right)^n = (d+d')(d+d'-1) \dots (d+d'-n+1),$

the equation may be transformed into

$$\{(d+d')^n + q_1(d+d')^{n-1} + \dots + q_n\} z = 0,$$

$$\text{or } (d+d'-a_1)(d+d'-a_2) \dots (d+d'-a_n) z = 0;$$

$$\therefore\ z = e^{a_1 t} e^{-td'} f_1(v) + e^{a_2 t} e^{-td'} f_2(v) + \dots + e^{a_n t} e^{-td'} f_n(v)$$

$$= x^{a_1} f_1(v-t) + x^{a_2} f_2(v-t) + \text{\&c.}$$

$$= x^{a_1} f_1\left(\frac{y}{x}\right) + x^{a_2} f_2\left(\frac{y}{x}\right) + \dots + x^{a_n} f_n\left(\frac{y}{x}\right).$$

119. Other substitutions either for the dependent variable or for the independent variables, that may be sometimes employed, are the following.

Ex. 1. $P\dfrac{dz}{dx}+Q\dfrac{dz}{dy}=Rf(z)$ where P, Q, and R are functions of x and y.

Let v denote a function of z determined by the equation

$$\frac{dv}{dz}=\frac{1}{f(z)};$$

then the equation becomes

$$P\frac{dv}{dx}+Q\frac{dv}{dy}=R.$$

Thus let the equation be

$$(y^2+mxy)\left(\frac{dz}{dx}-m\frac{dz}{dy}\right)=(max+by)(z^2+1),$$

then $v=\tan^{-1}z$, and the solution of

$$\frac{dv}{dx}-m\frac{dv}{dy}=\frac{max+by}{y^2+mxy}=\frac{a}{y}-\frac{a-b}{y+mx}$$

$$\text{is}\quad v=\phi(y+mx)-\frac{a}{m}\log y-\frac{(a-b)x}{y+mx};$$

$$\therefore\quad z=\tan\left\{\phi(y+mx)-\frac{a}{m}\log y-\frac{(a-b)x}{y+mx}\right\}.$$

Ex. 2. $\dfrac{1}{x^{n-1}}\dfrac{dz}{dx}+\dfrac{m}{y^{n-1}}\dfrac{dz}{dy}=\dfrac{az}{x^n}+\dfrac{bz}{y^n}+\dfrac{cz}{(xy)^{\frac{n}{2}}}$,

$$\text{let}\quad x^n=t,\quad y^n=v;$$

$$\therefore\quad \frac{dz}{dt}+m\frac{dz}{dv}=\left(\frac{a}{nt}+\frac{b}{nv}+\frac{c}{n\sqrt{tv}}\right)z;$$

$$\therefore\quad z=t^{\frac{a}{n}}\,.\,v^{\frac{b}{mn}}\,.\left(\sqrt{t}+\sqrt{\frac{1}{m}v}\right)^{\frac{2c}{n\sqrt{m}}}\phi(v-mt);$$

$$\text{or}\quad z=x^a y^{\frac{b}{m}}\left(x^{\frac{n}{2}}+\frac{1}{\sqrt{m}}y^{\frac{n}{2}}\right)^{\frac{2c}{n\sqrt{m}}}\phi(y^n-mx^n).$$

120. Since when operating relative to x we may treat y and the symbol $\dfrac{d}{dy}$ as simple constants, and *vice versa* when operating

relative to y, we may often solve a partial differential equation by integrating it with respect to one of the independent variables, and introducing arbitrary functions of the other instead of constants; and then effecting the operations indicated relative to the second variable, and interpreting the result in the usual way, which will be attended with no difficulty when only the exponential symbols $e^{a\frac{d}{dx}}$ or $e^{a\frac{d}{dy}}$ are involved.

Ex. 1. $\dfrac{d^2z}{dx^2} - a^2\dfrac{d^2z}{dy^2} = 0$; this may be written

$$\frac{d^2z}{dx^2} - (ad')^2 z = 0, \text{ or } \left(\frac{d}{dx} - ad'\right)\left(\frac{d}{dx} + ad'\right)z = 0;$$

$$\therefore\ z = e^{axd'}\phi(y) + e^{-axd'}\psi(y) = \phi(y+ax) + \psi(y-ax).$$

Hence $x\dfrac{d^2z}{dx^2} + 2\dfrac{dz}{dx} = a^2x\dfrac{d^2z}{dy^2}$,

which may be written $\dfrac{d^2(xz)}{dx^2} - a^2\dfrac{d^2(xz)}{dy^2} = 0$, gives

$$xz = \phi(y+ax) + \psi(y-ax).$$

Ex. 2. $x^4\dfrac{d^2z}{dx^2} - a^2\dfrac{d^2z}{dy^2} = 0$, or $x^4\dfrac{d^2z}{dx^2} - (ad')^2 z = 0$;

$$\therefore\ \frac{z}{x} = e^{\frac{a}{x}d'}\phi(y) + e^{-\frac{a}{x}d'}\psi(y) = \phi\left(y + \frac{a}{x}\right) + \psi\left(y - \frac{a}{x}\right).$$

(Ex. 2, Art. 69.)

Ex. 3. $\dfrac{d^2z}{dx^2} + a\dfrac{d^2z}{dxdy} - \dfrac{2z}{x^2} = 0$,

$$\text{or } \frac{d^2z}{dx^2} + ad'\,.\frac{dz}{dx} - \frac{2z}{x^2} = 0;$$

$$\therefore\ z = \left(1 - \frac{2}{axd'}\right)\phi'(y) + \left(1 + \frac{2}{axd'}\right)e^{-axd'}\psi'(y), \text{ (Ex. 6, Art. 74)}$$

$$= \phi'(y) - \frac{2}{ax}\phi(y) + \psi'(y-ax) + \frac{2}{ax}\psi(y-ax).$$

Ex. 4. $\frac{d^2z}{dx^2} - \frac{m(m+1)}{x^2} z - a^2 \frac{d^2z}{dy^2} = 0,$

$$\text{or } \frac{d^2z}{dx^2} - \frac{m(m+1)}{x^2} z - (ad')^2 z = 0\,;$$

$$\therefore\ zx^{m+1} = \left(x^3 \frac{d}{dx}\right)^m \frac{e^{axd'} \phi(y) + e^{-axd'} \psi(y)}{x^{2m-1}}, \text{ (Ex. 4, Art. 74)}$$

$$= \left(x^3 \frac{d}{dx}\right)^m \frac{\phi(y+ax) + \psi(y-ax)}{x^{2m-1}}.$$

Ex. 5. $\frac{d^2z}{dx^2} - \frac{2m}{x}\frac{dz}{dx} - a^2 \frac{d^2z}{dy^2} = 0,$

$$xz = \left(x^3 \frac{d}{dx}\right)^m \frac{\phi(y+ax) + \psi(y-ax)}{x^{2m-1}}. \quad \text{(Ex. 3, Art. 74.)}$$

Ex. 6. $x^2 \frac{d^2z}{dx^2} - 2mx \frac{dz}{dx} + 2mz - q^2x^2 \frac{d^2z}{dy^2} = 0,$

$$x^2z = \left(x^3 \frac{d}{dx}\right)^m \frac{\phi(y+qx) + \psi(y-qx)}{x^{2m-2}}. \quad \text{(Ex. 2, Art. 74.)}$$

If $m = -1$, in Ex. 4, we get the solution of

$$\frac{d^2(xz)}{dx^2} - \frac{2xz}{x^2} - q^2 \frac{d^2(xz)}{dy^2} = 0,$$

$$\text{or } x^2 \frac{d^2z}{dx^2} + 2x \frac{dz}{dx} - 2z - q^2x^2 \frac{d^2z}{dy^2} = 0, \text{ viz.}$$

$$x^2z = qx\{\phi'(y+qx) - \psi'(y-qx)\} - \phi(y+qx) - \psi(y-qx).$$

Ex. 7. $\frac{d^2z}{dx\,dy} + \frac{1}{x+y}\left(\frac{dz}{dx} + \frac{dz}{dy}\right) - \frac{2z}{(x+y)^2} = 0.$

Make $y + x = t$, $y - x = v$, then

$$t^2 \frac{d^2z}{dt^2} + 2t \frac{dz}{dt} - 2z - t^2 \frac{d^2z}{dv^2} = 0,$$

$$\therefore\ t^2z = t\phi'(t+v) - t\psi'(v-t) - \phi(t+v) - \psi(v-t)\,;$$

$$\therefore\ z(x+y)^2 = (x+y)\{\phi'(2y) - \psi'(-2x)\} - \phi(2y) - \psi(-2x).$$

Ex. 8. $\dfrac{d^2z}{dx^2} + \dfrac{2n}{x}\dfrac{dz}{dx} - a^2\dfrac{d^2z}{dy^2} = 0.$ (Ex. 3. Art. 74.)

$$x^{2n}z = \frac{d}{dx}\left\{\frac{1}{x}\left(x^3\frac{d}{dx}\right)^n \frac{\phi(y+ax) + \psi(y-ax)}{x^{2n-1}}\right\}. \quad \text{(Art. 65.)}$$

Hence if $n = 1$, we get, as in Ex. 1,

$$xz = \phi(y+ax) + \psi(y-ax);$$

and if $n = 2$, we find

$$x^3z = \phi(y+ax) + \psi(y-ax) - ax\phi'(y+ax) + ax\psi'(y-ax).$$

Ex. 9. $x^2\dfrac{d^2z}{dx^2} + qx^2\dfrac{d^2z}{dxdy} - 6z = 0.$ (Ex. 6. Art. 74.)

$$z = \frac{12}{q^2x^2}\phi(y) - \frac{6}{qx}\phi'(y) + \phi''(y)$$

$$+ \frac{12}{q^2x^2}\psi(y-qx) + \frac{6}{qx}\psi'(y-qx) + \psi''(y-qx).$$

Ex. 10. $\left(x\dfrac{d}{dx} + y\dfrac{d}{dy}\right)^2 z - 2\left(x\dfrac{d}{dx} + y\dfrac{d}{dy}\right)z + 2z = x^my^n.$

$$z = \frac{x^my^n}{(m+n-1)(m+n-2)} + x\phi\left(\frac{y}{x}\right) + x^2\psi\left(\frac{y}{x}\right).$$

Ex. 11. $f(x)\cdot\dfrac{d^2z}{dx^2} + \tfrac{1}{2}f'(x)\cdot\dfrac{dz}{dx} - a^2\dfrac{d^2z}{dy^2} = 0,$ (Ex. 1. Art. 48)

$$z = \phi\{y + a\int dx\,(fx)^{-\frac{1}{2}}\} + \psi[y - a\int dx\,\{f(x)\}^{-\frac{1}{2}}],$$

where $f(x)$ denotes a given function of x.

Ex. 12. $\dfrac{d^2z}{dx^2} - a^2x^{-\frac{4}{3}}\dfrac{d^2z}{dy^2} = 0.$

$$z = x^{\frac{1}{3}}\{\phi'(y + 3ax^{\frac{1}{3}}) + \psi'(y - 3ax^{\frac{1}{3}})\}$$

$$- \frac{1}{3a}\{\phi(y + 3ax^{\frac{1}{3}}) - \psi(y - 3ax^{\frac{1}{3}})\}.$$

Ex. 13. $\dfrac{d^2z}{dx^2} + f(x)\cdot\dfrac{d^2z}{dxdy} - \dfrac{f(x)}{x}\dfrac{dz}{dy} = 0,$ (Ex. 2. Art. 48)

$$z = x\int dx\,\frac{1}{x^2}\,\phi\{y - \int dx f(x)\} + x\psi(y).$$

Partial Differential Equations of a higher degree than the first.

121. To integrate $F(x, y, z, p, q) = 0$, when it contains terms of more than one dimension in p and q.

In order that $dz = pdx + qdy$ may be a perfect differential, we must have

$$\frac{dp}{dy} + q\frac{dp}{dz} = \frac{dq}{dx} + p\frac{dq}{dz}, \quad (2).$$

If from the proposed equation we determine

$$q, \frac{dq}{dx}, \frac{dq}{dz},$$

equation (2) becomes an equation for the determination of p of the form

$$\frac{dp}{dx} + M\frac{dp}{dy} + N\frac{dp}{dz} = L,$$

the integration of which depends on the integration of one of the equations

$$dp - Ldx = 0, \quad dy - Mdx = 0, \quad dz - Ndx = 0.$$

Let $p = f(x, y, z, a)$ be found from these equations, a being an arbitrary constant. This value of p and the corresponding value of q found from the proposed equation, being substituted in $dz = pdx + qdy$ render it an exact differential; and thus a value of z will be obtained involving two arbitrary constants a and b; and this will consequently be the complete primitive. The general primitive may be obtained by putting $b = \psi(a)$, differentiating the equation with regard to a, and eliminating a. The result containing one arbitrary function is as general as any solution which the equation admits.

Ex. 1. $p^2 + q^2 = 1$,

$$q = \sqrt{1-p^2}, \quad \frac{dq}{dx} = -\frac{p}{\sqrt{1-p^2}}\frac{dp}{dx}, \quad \frac{dq}{dz} = -\frac{p}{\sqrt{1-p^2}}\frac{dp}{dz};$$

$$\therefore \frac{dp}{dy} + \sqrt{1-p^2}\frac{dp}{dz} + \frac{p}{\sqrt{1-p^2}}\frac{dp}{dx} + \frac{p^2}{\sqrt{1-p^2}}\frac{dp}{dz} = 0,$$

$$\text{or } \frac{dp}{dz} + p\cdot\frac{dp}{dx} + \sqrt{1-p^2}\frac{dp}{dy} = 0;$$

which is integrable if we can integrate the system of equations

$$dp=0, \quad pdz-dx=0, \quad \sqrt{1-p^2}\,.\,dz-dy=0.$$

The first gives $p=a$, whence $q=\sqrt{1-a^2}$,

$$\therefore \ dz=adx+\sqrt{1-a^2}\,dy,$$

$$\text{or } z=ax+\sqrt{1-a^2}\,y+\phi(a).$$

Differentiating this with respect to a, we find

$$0=x-\frac{a}{\sqrt{1-a^2}}y+\phi'(a);$$

between which and the preceding we may eliminate a, when the form of the function $\phi(a)$ is assigned. This is a simple case of the Problem of finding surfaces of equivalent area to a given surface, that is, such that any cylindrical surface parallel to the axis of z, may always intercept equal areas in the required and given surface. If P, Q denote the values of $\frac{dz}{dx}$, $\frac{dz}{dy}$ in the given surface, the equation of condition is

$$p^2+q^2=P^2+Q^2;$$

which, if the given surface be a plane, leads to the equation of this example.

Ex. 2. $$p^m q^n = c^n.$$

The equation in p to be integrated is

$$\frac{dp}{dx}+\frac{n}{mc}p^{\frac{m}{n}+1}\frac{dp}{dy}+\left(\frac{n}{m}+1\right)p\frac{dp}{dz}=0;$$

$$\therefore \ dp=0, \quad \text{and } p=a^n, \quad \therefore \ q=ca^{-m},$$

$$\therefore \ dz=a^n dx+\frac{c}{a^m}dy,$$

$$\text{and} \quad z=a^n x+\frac{c}{a^m}y+\phi(a),$$

$$0 = na^{n-1}x - \frac{mc}{a^{m+1}}y + \phi'(a);$$

and it remains to eliminate a between these equations when the form of $\phi(a)$ is assigned.

If the proposed equation be

$$p^m q^n = c^n x^\alpha y^\beta z^\gamma,$$

and we take new independent variables x' and y' such that

$$\frac{dx'}{dx} = x^{\frac{\alpha}{m}}, \quad \frac{dy'}{dy} = y^{\frac{\beta}{n}}.$$

Then $$\left(\frac{dz}{dx'}\right)^m \cdot \left(\frac{dz}{dy'}\right)^n = c^n z^\gamma.$$

Now assume z' such that $\frac{dz'}{dz} = z^{\frac{-\gamma}{m+n}}$, then the equation becomes

$$\left(\frac{dz'}{dx'}\right)^m \cdot \left(\frac{dz'}{dy'}\right)^n = c^n,$$

and is reduced to the preceding.

Ex. 3. $$z = pq;$$

here, when the form of $\phi(a)$ is assigned, a must be eliminated between the equations,

$$\frac{z}{y+a} = x + \phi(a), \qquad \frac{z}{(y+a)^2} = -\phi'(a).$$

Thus if $\phi(a) = \frac{c}{a}$, then it will be found that $z = (c + \sqrt{xy})^2$.

Non-linear Equations of the Second and Higher Orders.

122. Here, besides the coefficients p and q of the first order which may be involved together with x, y, and z, the equation must contain one or more of the coefficients of the second order r, s, t; so that in its most general form it will be

$$F(x, y, z, p, q, r, s, t) = 0.$$

In partial differential equations of the second order, we cannot be certain of the form of the solution, nor pronounce before-

hand how many arbitrary functions it ought to contain. For let

$$u = f\{x, y, z, \phi(v), \psi(w)\} = 0,$$

be an equation containing two arbitrary functions of v and w two known functions of x, y and z; then p and q will be given by the equations

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0;$$

and r, s, and t by the equations

$$\frac{d^2u}{dx^2} = 0, \quad \frac{dd}{dxdy}u = 0, \quad \frac{d^2u}{dy^2} = 0.$$

These together with $u = 0$ make six equations, into which the six quantities

$$\phi(v), \quad \psi(w), \quad \phi'(v), \quad \psi'(w), \quad \phi''(v), \quad \psi''(w),$$

may enter. Consequently, it will not generally be possible to eliminate these six quantities and obtain a relation between x, y, z, p, q, r, s, t, independent of the forms of the functions ϕ and ψ; although particular cases do occur in which this can be effected.

In general, if $u = 0$ contain n independent variables and m functions of the form $\phi(v)$, where v is a determinate function of the variables; these functions can certainly be eliminated between the equations obtained after r differentiations, if

$$1 + n + \frac{n(n+1)}{1.2} + \ldots + \frac{n(n+1)\ldots(n+r-1)}{\lfloor r} > m(r+1),$$

the former quantity expressing the number of equations, and the latter the number of functions.

123. As an example of forming a partial differential equation of the second order, and of the sort we are now considering, by the elimination of two arbitrary functions, we may take

$$u = y - x\phi(z) - \psi(z) = 0;$$

$$\therefore \frac{du}{dx} = -\phi(z) - x\phi'(z)p - \psi'(z)p = 0,$$

$$\frac{du}{dy} = \quad 1 - x\phi'(z)q - \psi'(z)q = 0;$$

$$\therefore\ p + q\phi(z) = 0;$$

$$\therefore\ r + s\phi(z) + pq\phi'(z) = 0,$$

$$s + t\phi(z) + q^2\phi'(z) = 0;$$

$$\therefore\ qr - ps + (qs - pt)\,\phi(z) = 0,$$

$$\text{or}\ q^2r - 2pqs + p^2t = 0.$$

124. The equations

$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} + Q = 0,$$

$$\frac{d^2z}{dy^2} + P\frac{dz}{dy} + Q = 0,$$

where P and Q are functions of x, y and z, must be integrated as equations between two variables, y being regarded as constant in the former, and x in the latter; and arbitrary functions of those variables respectively, being introduced instead of constants.

Ex. $$(x - x^2)\frac{d^2z}{dx^2} + (y - xy - 2x)\frac{dz}{dx} - yz = 0,$$

$$(1 - x)\,z = x^{1-y}\phi(y) + \psi(y).$$

125. The equations

$$\frac{dd}{dx\,dy}z + P\frac{dz}{dx} = Q,$$

$$\frac{dd}{dx\,dy}z + P\frac{dz}{dy} = Q,$$

where P and Q do not contain z, are reducible to the case of Art. 105, by considering $\frac{dz}{dx}$ or $\frac{dz}{dy}$ respectively as a simple quantity v.

Ex. $$\frac{dd}{dx\,dy}z + \frac{y}{1-y^2}\frac{dz}{dx} = ay^3.$$

This being a linear equation in $\frac{dz}{dx}$ which is made integrable by the factor $\frac{1}{\sqrt{1-y^2}}$, we have

$$\frac{1}{\sqrt{1-y^2}}\,\frac{dz}{dx}=a\int dy\,\frac{y^3}{\sqrt{1-y^2}}=-a\left\{y^2\sqrt{1-y^2}+\tfrac{2}{3}(1-y^2)^{\frac{3}{2}}\right\}+\psi'(x)\,;$$

$$\therefore\ z=\{\psi(x)+\phi(y)\}\sqrt{1-y^2}-\frac{ax}{3}(2+y^2)(1-y^2).$$

126. Similarly, equations of the forms

$$f\left(x,\,y,\,z,\,\frac{dz}{dx},\,\frac{d^2z}{dx^2},\,\&c.,\,\frac{d^nz}{dx^n}\right)=0,$$

$$f\left(x,\,y,\,z,\,\frac{dz}{dy},\,\frac{d^2z}{dy^2},\,\&c.,\,\frac{d^nz}{dy^n}\right)=0,$$

may be treated as if they contained only two variables; arbitrary functions of y instead of constants being introduced into the solution of the first, and arbitrary functions of x into the solution of the second. To this case may also be reduced the equation

$$f\left(x,\,y,\,\frac{d^nz}{dy^n},\,\frac{dd^nz}{dxdy^n},\,\frac{d^2d^nz}{dx^2dy^n},\,\&c.,\,\frac{d^md^nz}{dx^mdy^n}\right)=0\,;$$

for by putting $\dfrac{d^nz}{dy^n}=v$, it becomes

$$f\left(x,\,y,\,v,\,\frac{dv}{dx},\,\frac{d^2v}{dx^2},\,\&c.,\,\frac{d^mv}{dx^m}\right)=0,$$

which will give a value of v containing m arbitrary functions of y; and then $\dfrac{d^nz}{dy^n}=v$ will give z involving n arbitrary functions of x.

127. To integrate the equation of the second order,

$$Rr+Ss+Tt=V,$$

where R, S, T, V are functions of x, y, z, p, q. The following process first given by Monge, may be frequently applied.

By means of the relations

$$dp=rdx+sdy,\quad dq=sdx+tdy,$$

eliminating two of the three coefficients, r and t, from the proposed, we get

$$Rdpdy+Tdqdx-Vdxdy=s\,\{R\,(dy)^2+T\,(dx)^2-Sdxdy\},$$

which is satisfied by

$$\left.\begin{aligned} Rdpdy + Tdqdx - Vdxdy = 0 \\ R(dy)^2 + T(dx)^2 - Sdxdy = 0 \end{aligned}\right\} \quad (1).$$

Let $M = a$, $N = b$, be two relations between x, y, z, p, q, and the arbitrary constants a, b, which satisfy these equations; then $M = \phi(N)$ satisfies the proposed equation. This will be shewn by proving that it can reproduce the proposed equation.

$$\text{Let } dy = mdx; \qquad \therefore\ Rm^2 - Sm + T = 0.$$

For each root of this equation, we have

$$dy - mdx = 0, \qquad Rmdp + Tdq - Vmdx = 0;$$

$$\left.\begin{aligned} \therefore\ dy &= mdx, \\ dq &= \frac{Vm}{T}dx - \frac{Rm}{T}dp, \\ dz &= pdx + qdy. \end{aligned}\right\} \quad (2)$$

Hence $M = a$ gives on differentiation

$$0 = \frac{dM}{dx} \cdot dx + \frac{dM}{dy} \cdot mdx + \frac{dM}{dz}(pdx + qmdx)$$

$$+ \frac{dM}{dp} \cdot dp + \frac{dM}{dq} \cdot \left(\frac{Vm}{T}dx - \frac{Rm}{T}dp\right),$$

wherein all the known relations (2) having been introduced, dx and dp must be independent,

$$\therefore\ 0 = \frac{dM}{dx} + m\frac{dM}{dy} + \frac{dM}{dz} \cdot (p + mq) + \frac{Vm}{T}\frac{dM}{dq},$$

$$0 = \frac{dM}{dp} - \frac{Rm}{T}\frac{dM}{dq};$$

$$\therefore\ \frac{dM}{dx} = -\left\{m\frac{dM}{dy} + (p + mq)\frac{dM}{dz} + \frac{Vm}{T}\frac{dM}{dq}\right\}.$$

$$\frac{dM}{dp} = \frac{Rm}{T}\frac{dM}{dq}.$$

So $$\frac{dN}{dx} = -\left\{m\frac{dN}{dy} + (p+mq)\frac{dN}{dz} + \frac{Vm}{T}\frac{dN}{dq}\right\},$$

$$\frac{dN}{dp} = \frac{Rm}{T}\frac{dN}{dq}.$$

By differentiating the assumed equation $M = \phi(N)$ we have

$$dM = \phi'(N) \,.\, dN.$$

Now

$$dM = -\left\{m\frac{dM}{dy} + (p+mq)\frac{dM}{dz} + \frac{Vm}{T}\frac{dM}{dq}\right\}dx + \frac{dM}{dy}dy$$

$$+\frac{dM}{dz}(pdx+qdy) + \frac{Rm}{T}\frac{dM}{dq}\,.\,dp + \frac{dM}{dq}\,.\,dq$$

$$= \left(\frac{dM}{dy} + q\frac{dM}{dz}\right)(dy - mdx) + \frac{1}{T}\frac{dM}{dq}(Rmdp + Tdq - Vmdx),$$

and a similar value exists for dN;

$$\therefore \left(\frac{dM}{dy} + q\frac{dM}{dz}\right)(dy - mdx) + \frac{1}{T}\frac{dM}{dq}(Rmdp + Tdq - Vmdx)$$

$$= \phi'(N)\left\{\left(\frac{dN}{dy} + q\frac{dN}{dz}\right)(dy - mdx)\right.$$

$$\left.+\frac{1}{T}\frac{dN}{dq}(Rmdp + Tdq - Vmdx)\right\};$$

which may be put under the form

$$Rmdp + Tdq - Vmdx = \omega\,(dy - mdx),$$

or $Rm\,(rdx+sdy) + T\,(sdx+tdy) - Vmdx = \omega\,(dy - mdx)$,

where dx and dy are independent;

$$\therefore\ Rmr + Ts - Vm = -\omega m,$$

$$Rms + Tt = \omega\,;$$

$$\therefore\ Rr + Ss + Tt = V - \frac{s}{m}(Rm^2 - Sm + T) = V.$$

Hence, $M = \phi(N)$ satisfies the proposed equation.

According as the roots of

$$Rm^2 - Sm + T = 0,$$

are unequal or equal, we are thus supplied with a total or partial differential equation for the determination of z.

As the reducing equations (1) may contain

$$x, y, z, p, q,$$

and as these together with $dz = pdx + qdy$ will generally lead to an equation containing three variables, which will not always admit of a single primitive (Art. 98), it may happen that the first integral of the proposed equation cannot be determined; but we must not thence conclude that the proposed equation does not admit of being solved.

128. Hence, to integrate the equation of the second order

$$Rr + Ss + Tt = V,$$

the process is to obtain a value of m from the equation

$$Rm^2 - Sm + T = 0,$$

to substitute it in the system

$$dy - mdx = 0, \qquad Rmdp + Tdq = Vmdx, \qquad (1)$$

to satisfy these conjointly or separately, by two relations between $x, y, z, p, q,$

$$M = a, \quad N = b,$$

then to put $M = \phi(N)$ which will be a first integral of the proposed, and to integrate this equation of the first order. But this determination of the second integral from the first will often be attended with great difficulty, on account of its involving an arbitrary function; and therefore when possible it is often more convenient to find from the second value of m another first integral of the form $M' = \psi(N')$, and between these to eliminate p or q, so as to obtain an equation involving only one differential coefficient, and which is therefore easily integrated.

129. If R, S, T be constant, and V a function of x and y only, then the values of m will be numerical, m and n suppose; and the integrals of equations (1) will be

$$y - mx = a, \quad Rmp + Tq = m\int dx V + b,$$

where, previous to integration, $mx+a$ is substituted for y in V, and after integration the value of a, viz. $y-mx$, is restored: consequently calling this value V_1, since $Rmn=T$, we have a first integral of the proposed, viz.

$$Rp+Rnq=V_1+\phi'(y-mx).$$

Next to integrate this equation of the first order, we have the reducing equations

$$dy-ndx=0,$$

$$Rdz-\{V_1+\phi'(y-mx)\}\,dx=0;$$

$$\therefore\ y-nx=\alpha,$$

$$Rz=\int dx\,V_1+\int dx\phi'(y-mx)+\beta,$$

$nx+\alpha$ being substituted for y before the integration is performed, and afterwards the value of α, $y-nx$, restored; this gives $\int dx\,V_1=V_2$, suppose,

and $\int dx\phi'(y-mx)=\int dx\phi'\{(n-m)x+\alpha\}=\dfrac{1}{n-m}\phi(y-mx)$;

hence, including the constant multiplier under the sign of the function, we have the complete integral required,

$$Rz=V_2+\phi(y-mx)+\psi(y-nx);$$

which agrees with the result obtained at once by separation of symbols from the proposed when put under the form

$$R\left(\frac{d}{dx}+m\frac{d}{dy}\right)\left(\frac{d}{dx}+n\frac{d}{dy}\right)z=f(x,y).$$

Ex. 1. $\dfrac{d^2z}{dx^2}-c^2\dfrac{d^2z}{dy^2}=0$, or $r-c^2t=0$.

The two systems of reducing equations here are

$$\left.\begin{aligned}dy+cdx=0\\dp+cdq=0\end{aligned}\right\},\qquad \left.\begin{aligned}dy-cdx=0\\dp-cdq=0\end{aligned}\right\};$$

from the former we get

$$y+cx=a,\quad p+cq=b;\quad \therefore\ p+cq=2c\phi'(y+cx);$$

similarly the latter gives $p-cq=-2c\psi'(y-cx)$;

hence, eliminating q, we find

$$p=\frac{dz}{dx}=c\phi'(y+cx)-c\psi'(y-cx),$$

$$\therefore\ z=\phi(y+cx)+\psi(y-cx).$$

Ex. 2. $r - t + \dfrac{4p}{x+y} = 0.$

Here the auxiliary equations are

$$dy - dx = 0, \quad dp - dq + \frac{4p}{x+y}\,dx = 0,$$

$$dy + dx = 0, \quad dp + dq - \frac{4p}{x+y}\,dx = 0 \quad (1).$$

From the former we find

$$y - x = a, \text{ and therefore } dp - dq + \frac{4pdx}{2y-a} = 0,$$

$$\text{or } (dp - dq)(2y - a) + (p - q)\,2dy + 2dz = 0,$$

$$\text{since } dz = pdx + qdy, \text{ and } y - x = a;$$

$$\therefore\ (2y - a)(p - q) + 2z = b = f(y - x),$$

$$\text{or } p - q + \frac{2z}{x+y} = \frac{f(y-x)}{x+y}.$$

But from the first of equations (1) we have $y + x = a'$;

$$\therefore\ \frac{dz}{dy} - \frac{dz}{dx} - \frac{2z}{a'} = -\frac{f(y-x)}{a'};$$

which, being a linear equation, may be integrated, and we get (Art. 113)

$$z = -e^{\frac{2y}{x+y}} \int dy e^{-\frac{2y}{a'}} \cdot \frac{f(y-x)}{a'} + e^{\frac{2y}{x+y}} \phi(x+y);$$

where, after the integration, a' is to be replaced by $x + y$.

Ex. 3. $$r - \frac{2p}{q}s + \frac{p^2}{q^2}t = 0,$$

$$\text{or } \quad rdxdy - \frac{2p}{q}sdxdy + \frac{p^2}{q^2}tdydx = 0,$$

or, eliminating r and t,

$$dpdy - s(dy)^2 - \frac{2p}{q}sdxdy + \frac{p^2}{q^2}\{dqdx - s(dx)^2\} = 0;$$

this resolves itself into

$$(dy)^2 + 2\frac{p}{q}dydx + \frac{p^2}{q^2}(dx)^2 = 0, \quad dpdy + \frac{p^2}{q^2}dqdx = 0;$$

whence arise the auxiliary equations

$$dy = -\frac{p}{q}\,dx, \quad dp - \frac{p}{q}\,dq = 0\,;$$

$$\text{but } dz = pdx + qdy = 0, \quad \therefore\ z = a,$$

$$\text{and } \frac{qdp - pdq}{q^2} = 0 \text{ gives } \frac{p}{q} = b\,;$$

$$\therefore\ p - qf(z) = 0, \text{ the first integral.}$$

To integrate this equation of the first order, we have

$$dz = 0, \qquad \text{or } z = \alpha,$$

$$dy + f(z)\,dx = 0, \qquad \text{or } y + xf(\alpha) = \beta\,;$$

$$\therefore\ y + xf(z) = \phi\,(z)\,;$$

this is the equation to the surface generated by a straight line, subjected to pass through three given fixed curves.

Ex. 4. $(1 + pq + q^2)\,r + (q^2 - p^2)\,s - (1 + pq + p^2)\,t = 0,$

$$\text{or } (1 + q\alpha)\,r + (q - p)\,\alpha s - (1 + p\alpha)\,t = 0,$$

putting $p + q = \alpha$. The values of m are to be obtained from the equation

$$(1 + q\alpha)\,m^2 - (q - p)\,\alpha m - 1 - p\alpha = 0,$$

which gives

$$m = -\frac{1 + p\alpha}{1 + q\alpha}\,,\ m = 1\,;$$

and the two systems of auxiliary equations answering to these values of m, are

$$dy - dx = 0, \quad (1 + q\alpha)\,dp - (1 + p\alpha)\,dq = 0,$$

$$(1 + q\alpha)\,dy + (1 + p\alpha)\,dx = 0, \quad dp + dq = 0. \quad (1)$$

The latter gives $p + q = b$, or $\alpha = b$,

$$x + y + bz = a\,; \qquad \therefore\ x + y + (p + q)\,z = \phi\,(p + q).$$

The former system gives $y - x = a_1$; and if we assume $p - q = \beta$, the second equation of this system becomes

$$(\alpha^2 + 2)\,d\beta - \alpha\beta d\alpha = 0, \quad \text{whence } \beta = b_1\,.\,(\alpha^2 + 2)^{\frac{1}{2}}\,;$$

$$\therefore\ p - q = \{(p + q)^2 + 2\}^{\frac{1}{2}}\,\psi\,(y - x).$$

Next to integrate this equation of the first order, since

$$p = \tfrac{1}{2}(\alpha + \beta), \quad q = \tfrac{1}{2}(\alpha - \beta),$$

we have

$$dz = \tfrac{1}{2}\alpha\,(dx + dy) + \tfrac{1}{2}\beta\,(dx - dy)$$
$$= \tfrac{1}{2}\alpha\,(dx + dy) + \tfrac{1}{2}(dx - dy)\,(\alpha^2 + 2)^{\frac{1}{2}}\,\psi\,(y - x),$$

this is integrable if we suppose α to be constant, and gives

$$z + \phi(\alpha) = \tfrac{1}{2}\alpha \,.\, (x + y) + (\alpha^2 + 2)^{\frac{1}{2}}\,\psi_1\,(y - x)\,;$$

which, combined with

$$\phi'(\alpha) = \tfrac{1}{2}(x + y) + \frac{\alpha}{\sqrt{\alpha^2 + 2}} \,.\, \psi_1\,(y - x),$$

represents the integral of the proposed equation.

Integration by a Series.

When other methods fail, partial differential equations, like equations between two variables, can be sometimes integrated by a series; and we shall now give an example taken from Euler of the process.

130. To integrate by a series the equation

$$bz + a\,(x + y)\,(p + q) + (x + y)^2\,s = 0.$$

Let $z = A\,(x+y)^m\phi(x) + B\,(x+y)^{m+1}\phi'(x) + C\,(x+y)^{m+2}\phi''(x) + \&c.;$

and as the integral must be symmetrical with respect to x and y, we must, after the determination of m, B, C, &c., replace $\phi(x)$, $\phi'(x)$, &c. by $\phi(x) + \psi(y)$, $\phi'(x) + \psi'(y)$, &c. Obtaining the values of p, q, s, and substituting them along with the assumed value of z in the proposed equation, we find

$$\begin{aligned} 0 &= \{b + 2ma + m\,(m - 1)\}\,A\,(x + y)^m\,\phi(x) \\ &+ \{bB + 2\,(m+1)\,aB + aA + (m+1)\,mB + mA\}\,(x + y)^{m+1}\phi'(x) \\ &+ \{bC + 2\,(m+2)\,aC + aB + (m+2)\,(m+1)\,C + (m+1)\,B\}\,(x+y)^{m+2}\phi''(x) \\ &+ \&c. \end{aligned}$$

which gives for the determination of m and B the relations

$$b + 2ma + m(m-1) = 0,$$

$$b + 2(m+1)a + (m+1)m = -(a+m)\frac{A}{B} \quad (1);$$

or, subtracting the former from the latter,

$$2a + 2m = -(a+m)\frac{A}{B}, \quad \therefore\ B = -\tfrac{1}{2}A.$$

For the determination of C we have

$$b + 2(m+2)a + (m+2)(m+1) = -(a+m+1)\frac{B}{C} = \tfrac{1}{2}(a+m+1)\frac{A}{C};$$

or, subtracting equation (1) from this,

$$2a + 2m + 2 = \tfrac{1}{2}(a+m+1)\frac{A}{C} - 2(a+m);$$

$$\therefore\ C = \frac{a+m+1}{2\,.\,2\,.\,(2a+2m+1)}A = \frac{(i-1)A}{2\,.\,2\,(2i-1)} \text{ putting } a+m=-i;$$

similarly $D = \dfrac{-(i-2)A}{2^2\,.\,2\,.\,3\,(2i-1)}$, $E = \dfrac{(i-2)(i-3)A}{2^3\,.\,2\,.\,3\,.\,4\,(2i-1)(2i-3)}$, &c.

The series will terminate whenever $a+m+n=n-i$ is a whole number, or when $b=(a+i)(a-i-1)$. Thus let $m=-a$, and therefore $b=a(a-1)$; it is evident, from the formation of the series, that $B = C = \&c. = 0$;

$$\therefore\ z = (x+y)^{-a}\{\phi(x) + \psi(y)\}.$$

Let $m = -a-2$, and therefore $b=(a+2)(a-3)$, then

$$z = \frac{\phi(x)+\psi(y)}{(x+y)^{a+2}} - \frac{1}{2}\,\frac{\phi'(x)+\psi'(y)}{(x+y)^{a+1}} + \frac{1}{12}\,\frac{\phi''(x)+\psi''(y)}{(x+y)^{a}}.$$

If $b=0$, the equation becomes

$$a(p+q) + (x+y)s = 0;$$

and we must have $a=-i$ or $=i+1$, and therefore $m=0$, or $m=-2i-1$; in the former case

$$z = \phi(x) + \psi(y) - \tfrac{1}{2}(x+y)\{\phi'(x)+\psi'(y)\}$$
$$+ \tfrac{1}{2}\frac{(i-1)(x+y)^2}{2(2i-1)}\{\phi''(x)+\psi''(y)\} - \&c.;$$

in the latter, we have $(x+y)^{2i+1}z$ equal to the same series.

To this form may likewise be reduced the equation (analogous to that of Riccati and integrable in the same cases)

$$(2m-1)^2 \frac{d^2z}{dx^2} = c^2 x^{\frac{4m}{1-2m}} \frac{d^2z}{dy^2},$$

by introducing new independent variables,

$$u = \tfrac{1}{2} x^{\frac{1}{1-2m}} + \frac{y}{2c}, \quad v = \tfrac{1}{2} x^{\frac{1}{1-2m}} - \frac{y}{2c};$$

when it becomes $(u+v)\dfrac{d^2z}{du\,dv} + m\dfrac{dz}{du} + m\dfrac{dz}{dv} = 0$; and hence z may be found in a series.

But when m is a positive integer, the solution may be readily obtained by assuming $x = t^{1\pm 2m}$; then

$$(2m \pm 1)^2 \frac{d^2z}{dx^2} = c^2 x^{\frac{-4m}{2m\pm 1}} \frac{d^2z}{dy^2}$$

is transformed into

$$\frac{dz^2}{dt^2} \mp \frac{2m}{t}\frac{dz}{dt} = c^2 \frac{d^2z}{dy^2}.$$

Hence, for the upper sign, we get (Ex. 5. Art. 120),

$$zt = \left(t^3 \frac{d}{dt}\right)^m \frac{\phi(y+ct) + \psi(y-ct)}{t^{2m-1}};$$

where, after performance of the operations indicated, we must replace t by $x^{\frac{1}{1+2m}}$; and for the lower sign the solution is (Art. 65)

$$zt^{2m} = \frac{d}{dt}\left\{\frac{1}{t}\left(t^3 \frac{d}{dt}\right)^m \frac{\phi(y+ct) + \psi(y-ct)}{t^{2m-1}}\right\};$$

where, after performing the operations indicated, we must replace t by $x^{\frac{1}{1-2m}}$. The latter result is obtained from observing that, if u be a solution of $\dfrac{d^2y}{dx^2} - \dfrac{2m}{x}\dfrac{dy}{dx} - c^2y = 0$, then $x^{-2m}\dfrac{du}{dx}$ is a solution of $\dfrac{d^2y}{dx^2} + \dfrac{2m}{x}\dfrac{dy}{dx} - c^2y = 0$.

Simultaneous Equations.

131. A system of partial differential equations which involve two dependent variables z and u, provided they are linear and have constant coefficients, may be treated in the same manner as simultaneous differential equations of the same description containing only one independent variable. For the two symbols of differentiation $\frac{d}{dx}$ and $\frac{d}{dy}$, since they do not affect each other, and obey in their combinations the same laws as ordinary algebraical quantities, may be treated as two independent constants.

Ex. 1.
$$\frac{dz}{dx} + a\frac{dz}{dy} + bz + c\frac{du}{dx} = 0,$$
$$\frac{du}{dx} + a\frac{du}{dy} + bu + c'\frac{dz}{dx} = 0.$$

Separating the symbols, and for convenience writing d and d' instead of $\frac{d}{dx}$ and $\frac{d}{dy}$, these become

$$(d + ad' + b)\,z + cdu = 0, \quad (d + ad' + b)\,u + c'dz = 0.$$

Now substitute in the former the value of u obtained from the latter, regarding d and d' as ordinary constants, and we get

$$\{(d + ad' + b)^2 - cc'd^2\}\,z = 0\,;$$

or, if we denote $1 - \sqrt{cc'}$ by $\frac{1}{m}$ and $1 + \sqrt{cc'}$ by $\frac{1}{n}$;

$$\{d + m\,(ad' + b)\}\,\{d + n\,(ad' + b)\}\,z = 0,$$

the integral of which is

$$z = e^{-mbx}\,\phi\,(y - amx) + e^{-nbx}\,\psi\,(y - anx),$$

where m and n have the values above-written; and from this u can be found.

Ex. 2. $$\frac{d^2z}{dxdy} + a\frac{du}{dy} = 0, \quad \frac{d^2u}{dxdy} + b\frac{dz}{dx} = 0\,;$$

$$\therefore\ \frac{d^3z}{dx^2dy} + a\frac{d^2u}{dxdy} = 0, \quad \text{or} \quad \frac{d^3z}{dx^2dy} - ab\frac{dz}{dx} = 0\,;$$

$$\therefore\ z = f(y) + x\phi\,(y) + e^{aby}\,\psi\,(x).$$

132. If $U=f(x,\ y,\ z,\ p,\ q)=0$ be a partial differential equation of the first order, the singular solution if there be one, as in the case of two variables, will be found by eliminating p and q between the three equations

$$U=0,\quad \frac{dU}{dp}=0,\quad \frac{dU}{dq}=0.$$

Ex. $(z-px-qy)^2-a^2(1+p^2+q^2)=0$, which expresses that the surface represented by the complete integral has the perpendicular from the origin on the tangent plane of a constant length. Here

$$\frac{dU}{dp}=-(z-px-qy)\,x-a^2p=0,$$

$$\frac{dU}{dq}=-(z-px-qy)\,y-a^2q=0;$$

eliminating p and q we find

$$x^2+y^2+z^2=a^2.$$

133. We shall terminate this part of the subject by the following geometrical problems; the first of them being the well-known solution by Monge of the Problem—to find a surface at every point of which the radii of curvature are equal and of the same sign. The conditions for this are expressed by the equations:

Ex. 1. $$\frac{p}{1+p^2}\,\frac{dp}{dx}=\frac{1}{q}\,\frac{dq}{dx};\qquad \frac{q}{1+q^2}\,\frac{dq}{dx}=\frac{1}{p}\,\frac{dp}{dy}.$$

Integrating these, and replacing the arbitrary constant by Y a function of y in the former, and by X a function of x in the latter, we find

$$1+p^2=q^2Y,\quad 1+q^2=p^2X;$$

$$\therefore\ p=\sqrt{\frac{Y+1}{XY-1}},\quad q=\sqrt{\frac{X+1}{XY-1}}\ \text{............}(1).$$

But as the object is to find z a function of x and y which shall satisfy the two proposed equations, the quantities p and q must by their nature satisfy the condition $\frac{dp}{dy}=\frac{dq}{dx}$, which becomes

$$\frac{1}{(1+X)^{\frac{3}{2}}}\,\frac{dX}{dx}=\frac{1}{(1+Y)^{\frac{3}{2}}}\,\frac{dY}{dy},$$

and is of the form $\phi(x) = \psi(y)$, whatever be the functions X and Y; it cannot therefore subsist for all values of x and y which are variables independent of one another, unless each member is reduced to the same arbitrary constant, $\frac{1}{2C}$ suppose; we then have

$$\frac{2C}{(1+X)^{\frac{3}{2}}}\frac{dX}{dx} = 1, \quad \frac{2C}{(1+Y)^{\frac{3}{2}}}\frac{dY}{dy} = 1,$$

which give by integration, a and b being two new arbitrary constants,

$$\frac{C}{\sqrt{1+X}} = a - x, \quad \frac{C}{\sqrt{1+Y}} = b - y.$$

Hence the quantities X and Y become known, and then p and q may be expressed in terms of x and y from equations (1), and substituting these values of p and q in $dz = pdx + qdy$ and integrating, we get

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = C^2.$$

Ex. 2. To find the surface which cuts at right angles the surfaces represented by the equation $xy + xz + yz = a^2$, when a assumes all values.

Here $p = -\frac{y+z}{x+y}$; $q = -\frac{x+z}{x+y}$, and the equation of condition is

$$1 + p\frac{dz}{dx} + q\frac{dz}{dy} = 0;$$

$$\therefore \ (y+z)\frac{dz}{dx} + (x+z)\frac{dz}{dy} = x + y;$$

the integral of which is

$$(x+y+z) \cdot (y-z)^2 = \phi\{(x+y+z) \cdot (x-z)^2\}.$$

Ex. 3. To find the surface which cuts at right angles all spheres that touch a plane in a given point; taking the given point for the origin, and the plane for that of xy, the equation is

$$2rz - z^2 = x^2 + y^2;$$

$$\therefore\ p = \frac{x}{r - z}, \qquad q = \frac{y}{r - z};$$

and the equation of condition is

$$x\frac{dz}{dx} + y\frac{dz}{dy} = z - r,$$

$$\text{or } 2xz\frac{dz}{dx} + 2yz\frac{dz}{dy} = z^2 - x^2 - y^2.$$

$$\text{Hence } ydx - xdy = 0, \quad \text{or } \frac{x}{y} = a,$$

$$2yzdz - (z^2 - x^2 - y^2)\,dy = 0,$$

$$\text{or } \frac{2zdz}{y} - \frac{z^2dy}{y^2} + (1 + a^2)\,dy = 0;$$

$$\therefore\ \frac{z^2}{y} + (1 + a^2)\,y = b; \quad \therefore\ z^2 + x^2 + y^2 = y\phi\left(\frac{x}{y}\right).$$

If the required surface is to be of the second order, we must have

$$\phi\left(\frac{x}{y}\right) = b\frac{x}{y} + c, \quad \therefore\ z^2 + x^2 + y^2 = bx + cy,$$

which represents spheres passing through the given point, and having their centres in the given plane.

By integrating the auxiliary equations

$$xdy - ydx = 0, \quad dx\,(r - z) + xdz = 0,$$

we get
$$x\,(r - z) = \phi\left(\frac{y}{x}\right).$$

And if we wish to determine the arbitrary function so that when $y = ax^2$, z shall equal $r + bx$, then

$$x\,(z - r) = \frac{b}{a^2}\left(\frac{y}{x}\right)^2.$$

Fig. 1.

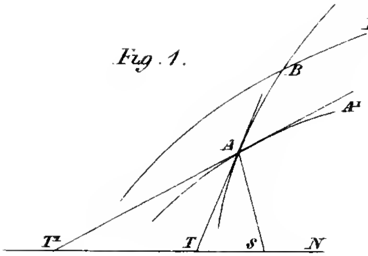


Fig. 2

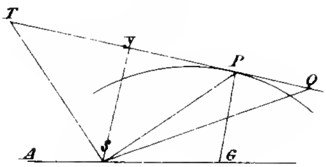


Fig. 3.

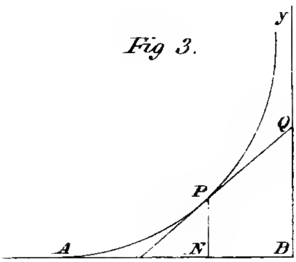


Fig. 4

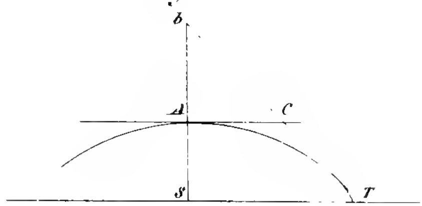


Fig. 6

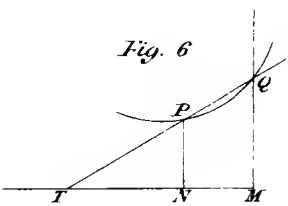


Fig. 5.

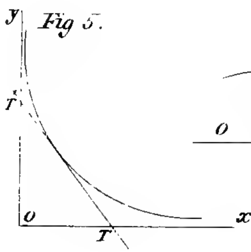


Fig. 7.

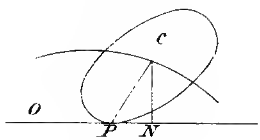
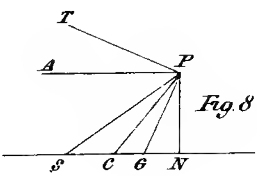


Fig. 8.



A TREATISE

ON THE

CALCULUS OF FINITE DIFFERENCES.

SECTION I.

DIRECT METHOD OF DIFFERENCES.

Definitions and Principles.

Art. 1. The Difference of a Function of one or more variables is the result obtained by subtracting from one another the two values of the function that arise from giving to the variables contained in it, different assigned values.

Thus if u_x denote any function of a variable x, and u_{x+h} the same function of $x+h$, and if the value of u_x be subtracted from that of u_{x+h}, the result is called the *Difference* of u_x; and the quantity h is called the *increment* of the principal variable x. In the Differential Calculus it is the first term only of the series, arranged according to ascending powers of h, expressing $u_{x+h}-u_x$, or rather the coefficient of h in that term, with which we are principally concerned, and which we usually write $h\dfrac{du_x}{dx}$. But in the Calculus of Finite Differences, it is the whole of that series which forms the object of our investigations, and it is usually written Δu_x, so that

$$\Delta u_x = u_{x+h} - u_x.$$

2. In deducing the Difference of any proposed function u_x, the increment of the principal variable x is always supposed to be known and finite; its value however is not commonly taken

to be an undetermined quantity h, but to be unity; both for the sake of simplicity, and because that is the value which the increment must necessarily have, in order to pass on to the succeeding term, when u_x is regarded as the general term of a Series; and it is in that light that it is by far the most frequently regarded in Finite Differences; so that the operation which the symbol Δ prefixed to u_x implies is, for the most part,

$$\Delta u_x = u_{x+1} - u_x.$$

There are, however, in this Subject, several important theorems which it is advantageous to investigate on the hypothesis of an indeterminate increment h for the principal variable, instead of unity; as the process is the same on either supposition, and the result one of greater generality. And in any case if it should be desirable to introduce the same hypothesis, the expression $f(x)$ must be prepared by first writing hz instead of x; then when z becomes $z+1$, hz will become $h(z+1)$ or $hz+h$, that is, the corresponding increment of x will be h; and if the requisite analytical operations be now performed upon $f(hz)$, on the usual supposition of $\Delta z = 1$, we shall get the result expressed in terms of x, and adapted to the supposition of an indeterminate increment for x, by restoring $\frac{x}{h}$ in the place of z.

3. By a Series is meant a regular progression of terms increasing or decreasing in magnitude according to a certain law; hence, when that law is given, and also the place of any term in the series, the magnitude of the term may be found, and thus the successive terms of the series may be produced in order. The place of any term in a series is assigned by giving the number of terms by which it is removed from some one which is considered as fixed. This number is called the *Index* of the term to which it belongs. Thus in the series

$$0,\ 1,\ 8,\ 27,\ 64,\ \ldots\ x^3 \ldots.$$

taking the first term as the point of departure, we have the corresponding series of indices

$$0,\ 1,\ 2,\ 3,\ 4,\ \ldots\ x,\ \ldots$$

If the series be continued backwards, the indices must be considered as negative; thus the backward continuation of the above series gives the terms

$$\ldots -x^3, \ldots -27, \; -8, \; -1,$$

with the corresponding indices

$$-x, \ldots -3, \; -2, \; -1.$$

4. Since the magnitude of every term is determined solely by its index and by the law of the series, it follows that any term is a certain function of its index, the form of which does not alter in passing from one term to another, but remains the same throughout the whole series. Thus in the above series every term is the cube of its index.

This function analytically expressed is called the general term of the series; (in the above series the general term is x^3;) and it is evident that all the terms of the series will be produced from it in order, by substituting successively for the index x, the progression of natural numbers

$$\ldots -2, \; -1, \; 0, \; 1, \; 2, \ldots$$

The general term of a series is usually denoted by u_x, where u_x is a certain function of x determined by the nature of the series. Thus, u_x denoting the general term, the series will be

$$\ldots u_{-2}, \; u_{-1}, \; u_0, \; u_1, \; u_2, \ldots u_{x-1}, \; u_x, \; u_{x+1}, \ldots$$

Any group of consecutive terms u_x, u_{x+1}, u_{x+2}, &c. are called successive values of the function u_x.

5. The excess of any term u_{x+1} above that which immediately precedes it, or the function $u_{x+1} - u_x$ is called, as has been stated, the Difference of the function u_x, and is denoted by Δu_x. (In certain cases, which will however be expressly mentioned, we shall take Δu_x to mean $u_{x+h} - u_x$.) Hence the characteristic or symbol of operation Δ, prefixed to a given function of x, denotes the series of operations of changing x into $x+1$, and of subtracting from the altered value of the function the proposed value.

It is obvious that $u_{x+1} - u_x$ is itself in general a certain function of x, the nature of which is entirely dependent on that of the original function u_x from which it is derived, and is susceptible of a difference.

The difference, consequently, of the function Δu_x (which must be considered as having Δu for its characteristic, in the same manner as u_x has u) is

$$\Delta\,(\Delta u_x) = \Delta u_{x+1} - \Delta u_x,$$

which is usually written $\Delta^2 u_x$.

In like manner

$$\Delta\,(\Delta^2 u_x) = \Delta^3 u_x = \Delta^2 u_{x+1} - \Delta^2 u_x,$$

$$\Delta\,(\Delta^3 u_x) = \Delta^4 u_x = \Delta^3 u_{x+1} - \Delta^3 u_x,$$

....

$$\Delta^n u_x = \Delta^{n-1} u_{x+1} - \Delta^{n-1} u_x.$$

6. Hence if in any function of x we change x into $x+1$, and from the result subtract the proposed function, we obtain the first difference of the proposed function; and the second, third, &c. differences are formed, each from the preceding, by a similar operation. To determine these differences of given functions, and to investigate the relations which hold between differences of any orders and the functions from which they are derived, is the object of the direct method of Finite Differences.

We shall now proceed to give instances of finding the differences of various functions, according to the above definition.

Differences of Explicit Functions.

7. To find the difference of $au_x + c$, u_x being any function of x, and a and c quantities independent of x.

$$\Delta\,(au_x + c) = au_{x+1} + c - (au_x + c) = a\,(u_{x+1} - u_x) = a\Delta u_x.$$

Hence, making $a = 0$, $\Delta c = 0$.

8. To find the difference of the sum of any number of functions of x.

$$\Delta\left(u_x+v_x+w_x\right)=u_{x+1}+v_{x+1}+w_{x+1}-\left(u_x+v_x+w_x\right)$$
$$=u_{x+1}-u_x+v_{x+1}-v_x+w_{x+1}-w_x$$
$$=\Delta u_x+\Delta v_x+\Delta w_x.$$

If some of the functions be preceded by negative signs, we shall find a similar result, viz.

$$\Delta\left(u_x-v_x-w_x\right)=\Delta u_x-\Delta v_x-\Delta w_x.$$

9. To find the difference of the product of two functions.

$$\Delta\left(u_x v_x\right)=u_{x+1}v_{x+1}-u_x v_x=\left(u_x+\Delta u_x\right)\left(v_x+\Delta v_x\right)-u_x v_x$$
$$=u_x\Delta v_x+v_x\Delta u_x+\Delta u_x\,.\,\Delta v_x=u_x\Delta v_x+v_{x+1}\Delta u_x.$$

10. To find the difference of the quotient of two functions.

$$\Delta\left(\frac{u_x}{v_x}\right)=\frac{u_{x+1}}{v_{x+1}}-\frac{u_x}{v_x}=\frac{\left(u_x+\Delta u_x\right)v_x-\left(v_x+\Delta v_x\right)u_x}{v_{x+1}v_x}$$
$$=\frac{v_x\Delta u_x-u_x\Delta v_x}{v_{x+1}v_x}.$$

11. To find the difference of the continued product of any number of successive values of a function.

$$\Delta\left(u_x u_{x+1}\ldots u_{x+n}\right)=u_{x+1}u_{x+2}\ldots u_{x+n+1}-u_x u_{x+1}\ldots u_{x+n}$$
$$=u_{x+1}u_{x+2}\ldots u_{x+n}\left(u_{x+n+1}-u_x\right).$$

Hence in the particular case where $u_x=a+bx$,

since $u_{x+n+1}=a+b\left(x+n+1\right)=u_x+\left(n+1\right)b$,

$$\Delta\left(u_x u_{x+1}\ldots u_{x+n}\right)=u_{x+1}u_{x+2}\ldots u_{x+n}\,.\,\left(n+1\right)b.$$

12. To find the difference of a fraction whose numerator and denominator are the continued products of any number of successive values of two functions u_x and v_x respectively.

$$\Delta\left(\frac{u_x u_{x+1}\ldots u_{x+n}}{v_x v_{x+1}\ldots v_{x+m}}\right)=\frac{u_{x+1}u_{x+2}\ldots u_{x+n}}{v_x v_{x+1}\ldots v_{x+m+1}}\left(v_x u_{x+n+1}-u_x v_{x+m+1}\right).$$

Hence $$\Delta \frac{1}{v_x v_{x+1} \dots v_{x+m}} = - \frac{v_{x+m+1} - v_x}{v_x v_{x+1} \dots v_{x+m+1}},$$

and in the particular case where $v_x = a + bx$, and therefore

$$v_{x+m+1} = a + b(x + m + 1) = v_x + (m+1)b,$$

$$\Delta \frac{1}{v_x v_{x+1} \dots v_{x+m}} = - \frac{(m+1)b}{v_x v_{x+1} \dots v_{x+m+1}}.$$

13. To find the differences of any rational integral function, and to shew that the n^{th} difference of a rational integral function of the n^{th} degree, is constant.

Let $u_x = Ax^n + Bx^{n-1} + \dots + Ix^2 + Kx + L$, be a rational integral function; then its first difference is

$$\Delta u_x = A\{(x+1)^n - x^n\} + B\{(x+1)^{n-1} - x^{n-1}\} + \dots + I(2x+1) + K$$

$$= nAx^{n-1} + B_1 x^{n-2} + \dots + I_1 x + K_1,$$

which is a rational integral function, one degree lower than the original function. In like manner, for the difference of this, or the second difference of u_x, we have

$$\Delta^2 u_x = n(n-1)Ax^{n-2} + B_2 x^{n-3} + \dots + I_2,$$

and so on; and for the n^{th} difference, we have

$$\Delta^n u_x = n(n-1) \dots 3 \,.\, 2 \,.\, 1 \,.\, A.$$

Hence the n^{th} difference is constant, and the differences of all orders superior to the n^{th} vanish.

Also $\Delta^n (x^n) = 1 \,.\, 2 \,.\, 3 \dots n.$

14. To find the differences of a^x.

$$\Delta a^x = a^{x+1} - a^x = a^x (a-1),$$

$$\Delta^2 a^x = (a-1)\,\Delta a^x = a^x (a-1)^2,$$

$$\Delta^n a^x = a^x (a-1)^n.$$

Also $\Delta a^{v_x} = a^{v_x + \Delta v_x} - a^{v_x} = a^{v_x}(a^{\Delta v_x} - 1).$

15. To find the difference of $\log v_x$.

$$\Delta(\log v_x) = \log v_{x+1} - \log v_x = \log \frac{v_{x+1}}{v_x} = \log\left(1 + \frac{\Delta v_x}{v_x}\right).$$

16. To find the differences of $\sin v_x$ and $\cos v_x$.

$$\Delta \sin v_x = \sin(v_x + \Delta v_x) - \sin v_x = 2 \sin \frac{\Delta v_x}{2} \cos\left(v_x + \frac{\Delta v_x}{2}\right)$$

$$= 2 \sin \frac{\Delta v_x}{2} \sin\{v_x + \tfrac{1}{2}(\pi + \Delta v_x)\}.$$

$$\Delta \cos v_x = \cos(v_x + \Delta v_x) - \cos v_x = -2 \sin \frac{\Delta v_x}{2} \sin\left(v_x + \frac{\Delta v_x}{2}\right)$$

$$= 2 \sin \frac{\Delta v_x}{2} \cos\{v_x + \tfrac{1}{2}(\pi + \Delta v_x)\}.$$

Hence making $v_x = x\theta + \alpha$ so that $\Delta v_x = \theta$, we get

$$\Delta \sin(x\theta + \alpha) = 2 \sin \tfrac{1}{2}\theta \sin\{x\theta + \alpha + \tfrac{1}{2}(\pi + \theta)\},$$

which shews that the difference of the sine of an angle $x\theta + \alpha$ is found by adding $\frac{1}{2}(\pi + \theta)$ to the angle, and multiplying by $2 \sin \frac{1}{2}\theta$.

Hence, repeating the operation n times, we get

$$\Delta^n \sin(x\theta + \alpha) = (2 \sin \tfrac{1}{2}\theta)^n \sin\{x\theta + \alpha + \tfrac{1}{2}n(\pi + \theta)\}.$$

Similarly,

$$\Delta^n \cos(x\theta + \alpha) = (2 \sin \tfrac{1}{2}\theta)^n \cos\{x\theta + \alpha + \tfrac{1}{2}n(\pi + \theta)\}.$$

Also $$\Delta^n \sin \pi x = 2^n \sin \pi(x + n),$$

$$\Delta^n \cos \pi x = 2^n \cos \pi(x + n).$$

16*. To find the differences of $\tan v_x$ and $\tan^{-1} v_x$.

$$\Delta \tan v_x = \tan v_{x+1} - \tan v_x = \frac{\sin v_{x+1} \cos v_x - \cos v_{x+1} \sin v_x}{\cos v_{x+1} \cos v_x}$$

$$= \frac{\sin(v_{x+1} - v_x)}{\cos v_{x+1} \cos v_x} = \frac{\sin \Delta v_x}{\cos v_{x+1} \cos v_x}.$$

Hence $$\Delta \tan x\theta = \frac{\sin\theta}{\cos(x+1)\theta\cos x\theta}.$$

Also $$\Delta \tan^{-1} v_x = \tan^{-1} v_{x+1} - \tan^{-1} v_x = \tan^{-1}\frac{v_{x+1}-v_x}{1+v_{x+1}v_x}$$

$$= \tan^{-1}\frac{\Delta v_x}{1+v_{x+1}v_x}.$$

Hence $$\Delta \tan^{-1} x\theta = \tan^{-1}\frac{\theta}{1+(x+1)x\theta^2}.$$

Relations between the Successive Values and the Differences of a Function.

17. The successive values of u_x any function of x are, as has been stated, the values which arise from substituting $x+1$, $x+2$, &c., or more generally $x+h$, $x+2h$, &c., for x in u_x. These values are usually written u_{x+h}, u_{x+2h}, u_{x+3h}, &c.; but it is further requisite that the operation of forming these values should be denoted by a prefixed symbol, in order that the notation for the successive values of u_x may be analogous to that for its successive differences, and in order that we may be able to avail ourselves of the advantages which, in these investigations, the Method of Separation of Symbols offers. Suppose therefore D to be a symbol of operation implying the change of x into $x+h$ in any function of x to which it is prefixed, so that $Du_x = u_{x+h}$; then

$$D(Du_x) = Du_{x+h} = u_{x+2h},$$

which may be written $D^2u_x = u_{x+2h}$; similarly $D^3u_x = u_{x+3h}$, and generally $D^nu_x = u_{x+nh}$.

Hence since $\Delta u_x = u_{x+h} - u_x$, we have $\Delta u_x = Du_x - u_x$, which, if we separate the symbol of operation D from that of quantity u_x, may be written $\Delta u_x = (D-1)u_x$, and expresses that the operation indicated by Δ is equivalent to that indicated by $D-1$. Also since $u_{x+1} = u_x + \Delta u_x$, we have $Du_x = u_x + \Delta u_x$, which, if we separate the symbols, may be written

$$Du_x = (1+\Delta)u_x,$$

and expresses that the operation indicated by D is equivalent to that denoted by $1+\Delta$.

18. To express $\Delta^n u_x$ by u_x and its n successive values,

$$u_{x+h},\quad u_{x+2h},\ \dots\ u_{x+nh}.$$

We here take h instead of unity for the increment of the principal variable, as the investigation is precisely the same on either supposition.

$$\Delta u_x = u_{x+h} - u_x,$$

$$\Delta^2 u_x = u_{x+2h} - u_{x+h} - (u_{x+h} - u_x) = u_{x+2h} - 2u_{x+h} + u_x,$$

$$\Delta^3 u_x = u_{x+3h} - 2u_{x+2h} + u_{x+h} - (u_{x+2h} - 2u_{x+h} + u_x),$$

$$= u_{x+3h} - 3u_{x+2h} + 3u_{x+h} - u_x.$$

Now suppose this law of the coefficients, which as far as we have gone is the same as that of an expanded binomial whose index is the order of the difference, to hold for the n^{th} difference, so that

$$\Delta^n u_x =$$

$$u_{x+nh} - p_1 u_{x+(n-1)h} + p_2 u_{x+(n-2)h} - \dots \mp p_1 u_{x+h} \pm u_x,$$

$$\text{then } \Delta^{n+1} u_x =$$

$$u_{x+(n+1)h} - p_1 u_{x+nh} + p_2 u_{x+(n-1)h} - \dots \mp p_1 u_{x+2h} \pm u_{x+h}$$

$$-\,(u_{x+nh} - p_1 u_{x+(n-1)h} + \dots \pm p_2 u_{x+2h} \mp p_1 u_{x+h} \pm u_x)$$

$$= u_{x+(n+1)h} - (1+p_1)\, u_{x+nh} + (p_1+p_2)\, u_{x+(n-1)h} - \dots$$

$$\pm (1+p_1)\, u_{x+h} \mp u_x,$$

which is the same alteration with regard to the coefficients as occurs in passing from $(z-1)^n$ to $(z-1)^{n+1}$. If therefore, for any value of n supposed a positive integer, the coefficients of the expansions of $\Delta^n u_x$ and $(z-1)^n$ are the same, they will always be the same; but these coefficients are identical, as we have seen, when $n=1, 2, 3$; therefore they are always the same;

$$\therefore\ \Delta^n u_x = u_{x+nh} - n u_{x+(n-1)h} + \frac{n(n-1)}{1.2} u_{x+(n-2)h} - \ldots$$

$$\mp n u_{x+h} \pm u_x;$$

or, supposing $h = 1$,

$$\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \ldots \mp n u_{x+1} \pm u_x.$$

19. If for the descending values u_{x+n}, u_{x+n-1}, &c. we substitute their equivalents (Art. 17), the second member becomes

$$D^n u_x - n D^{n-1} u_x + \frac{n(n-1)}{1.2} D^{n-2} u_x - \&c. \ldots \pm u_x,$$

or, separating the symbols of operation from those of quantity,

$$(D^n - n D^{n-1} + \frac{n(n-1)}{1.2} D^{n-2} - \&c. \ldots \pm 1)\, u_x,$$

$$\therefore\ \Delta^n u_x = (D-1)^n u_x,$$

each term of the development of $(D-1)^n$ being understood to be prefixed to u_x. This formula results immediately from the definitions of Δ and D (Art. 17); for as the operation denoted by Δ is equivalent to that denoted by $D-1$, if these operations be performed n times upon the same function u_x, we must have

$$\Delta^n u_x = (D-1)^n u_x.$$

20. If in the series just investigated, we assign a particular value to u_x, we shall readily obtain an expression for its n^{th} difference. Thus let $u_x = x^m$,

$$\therefore\ \Delta^n (x^m) = (x+n)^m - n(x+n-1)^m + \frac{n(n-1)}{1.2}(x+n-2)^m - \&c.,$$

in which equation, if $n > m$, since the former member vanishes (Art. 13), the second member is zero for every value of x; and if $m = n$, so that $\Delta^n x^n = 1.2.3 \ldots n$, we get

$$1.2.3 \ldots n = (x+n)^n - n(x+n-1)^n + \frac{n(n-1)}{1.2}(x+n-2)^n - \&c.;$$

and making $x=0$, since the equation holds for all values of x,

$$1.2.3\ldots n = n^n - n(n-1)^n + \frac{n(n-1)}{1.2}(n-2)^n - \&c.$$

If $x=0$, and $\Delta^n 0^m$ denote the particular value of $\Delta^n x^m$ when $x=0$, we have

$$\Delta^n 0^m = n^m - n(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \&c.$$

Of the numbers comprised in the form $\Delta^n 0^m$, which are called the Differences of zero, we shall make considerable use in future investigations; whenever $n > m$ the value is zero, in other cases it may be computed by the above formula; thus,

$$\Delta 0^m = 1,$$

$$\Delta^2 0^2 = 2, \qquad \Delta^2 0^3 = 6, \qquad \Delta^2 0^4 = 14, \ldots$$

$$\Delta^3 0^3 = 6, \qquad \Delta^3 0^4 = 36, \qquad \Delta^3 0^5 = 150, \ldots$$

21. Reversing the order of the series in Art. 18, we find

$$(-1)^n \Delta^n u_x = u_x - n u_{x+1} + \frac{n(n-1)}{1.2} u_{x+2} - \ldots \pm u_{x+n}.$$

Also putting $u_x = x^m$, and then supposing $m=n$, $x=1$, we find successively,

$$(-1)^n \Delta^n x^m = x^m - n(x+1)^m + \frac{n(n-1)}{1.2}(x+2)^m - \ldots \pm (x+n)^m,$$

$$(-1)^n 1.2.3..n = x^n - n(x+1)^n + \frac{n(n-1)}{1.2}(x+2)^n - \ldots \pm (x+n)^n,$$

$$(-1)^n 1.2.3..n = 1^n - n.2^n + \frac{n(n-1)}{1.2}.3^n - \ldots \pm (n+1)^n.$$

22. Hence it may be proved that $1.2.3\ldots(p-1)+1$ is divisible by p, if p be a prime number. (Wilson's Theorem)

Let $u_x = x^n - 1$;

$$\therefore \Delta^n(x^n-1) = 1.2.3\ldots n = (x+n)^n - 1 - n\{(x+n-1)^n - 1\}$$

$$+ \frac{n(n-1)}{1.2}\{(x+n-2)^n - 1\} - \&c.$$

Let $x=1$, and $n+1=p$, then $x+n=p$, and

$$1 \,.\, 2 \,.\, 3 \ldots (p-1)+1 = p^n - (p-1)\{(p-1)^{p-1}-1\}$$
$$+\frac{(p-1)(p-2)}{1 \,.\, 2}\{(p-2)^{p-1}-1\} - \&c.$$

Now by Fermat's Theorem every term of the second member is divisible by p when p is a prime number; consequently

$$1 \,.\, 2 \,.\, 3 \ldots (p-1)+1 \text{ is divisible by } p.$$

23. To express u_{x+nh} by u_x and its first n differences.

We take h instead of unity for the increment of the principal variable, the investigation being precisely the same on either supposition.

$$u_{x+h} = u_x + \Delta u_x,$$

$$u_{x+2h} = u_x + \Delta u_x + \Delta(u_x + \Delta u_x) = u_x + 2\Delta u_x + \Delta^2 u_x,$$

$$u_{x+3h} = u_x + 2\Delta u_x + \Delta^2 u_x + \Delta(u_x + 2\Delta u_x + \Delta^2 u_x)$$
$$= u_x + 3\Delta u_x + 3\Delta^2 u_x + \Delta^3 u_x.$$

Now suppose this law of coefficients, which as far as we have gone is the same as that of an expanded binomial whose index is the number of increments which the principal variable has received, to hold for n increments, so that

$$u_{x+nh} = u_x + p_1 \Delta u_x + p_2 \Delta^2 u_x + \ldots + p_1 \Delta^{n-1} u_x + \Delta^n u_x,$$

then $u_{x+(n+1)h} = u_x + p_1 \Delta u_x + p_2 \Delta^2 u_x + \ldots + p_1 \Delta^{n-1} u_x + \Delta^n u_x$

$$+\Delta(u_x + p_1 \Delta u_x + \ldots + p_2 \Delta^{n-2} u + p_1 \Delta^{n-1} u_x + \Delta^n u_x)$$
$$= u_x + (1+p_1)\Delta u_x + (p_1+p_2)\Delta^2 u_x + \ldots$$
$$+ (p_2+p_1)\Delta^{n-1} u_x + (p_1+1)\Delta^n u_x + \Delta^{n+1} u_x,$$

which is the same alteration with regard to the coefficients as occurs in passing from $(1+z)^n$ to $(1+z)^{n+1}$.

Hence if the coefficients of the expansions of u_{x+nh} and $(1+z)^n$ are the same for any value of n supposed a positive integer, they

will always be the same; but they are identical as we have seen when $n = 1, 2, 3$; therefore they are always the same;

$$\therefore\ u_{x+nh} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \ldots + n\Delta^{n-1}u_x + \Delta^n u_x,$$

or, supposing $h = 1$,

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \ldots + n\Delta^{n-1}u_x + \Delta^n u_x.$$

24. This result, if we separate the symbols, may be written

$$u_{x+n} = \left(1 + n\Delta + \frac{n(n-1)}{1.2}\Delta^2 + \ldots + n\Delta^{n-1} + \Delta^n\right) u_x,$$

$$\text{or } u_{x+n} = (1+\Delta)^n u_x,$$

each term of the development of $(1 + \Delta)^n$ being understood to be prefixed to u_x. The same formula follows immediately from the definitions of the symbols D and Δ; for as D is equivalent to $1 + \Delta$, if the operations denoted by D and $1 + \Delta$ be performed n times upon the same function u_x, we must have

$$D^n u_x = (1+\Delta)^n u_x; \text{ but } D^n u_x = u_{x+n},$$

$$\therefore\ u_{x+n} = (1+\Delta)^n u_x.$$

24*. Let $u_x = x^m$, then $u_{x+n} = (x+n)^m$;

$$\therefore\ (x+n)^m = (1+\Delta)^n x^m,$$

$$\text{and making } x = 0, \quad n^m = (1+\Delta)^n 0^m,$$

each term of the development of $(1 + \Delta)^n$ being prefixed, as said above, to x^m and 0^m, respectively. By the latter formula any power of a number is expressed by the numbers comprised in the form $\Delta^n 0^m$, i.e. by the differences of zero.

25. To deduce Taylor's Theorem from the formula

$$u_{x+nh} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2}\Delta^2 u_x + \&c.$$

Let $nh = t$, and let h be infinitely diminished whilst t remains finite; therefore n is infinitely increased; and since

h is indefinitely diminished, we have, regarding the differential coefficient as the limit of the ratio of the simultaneous increments of the function and the variable,

$$\frac{\Delta u_x}{h} = \frac{du_x}{dx},$$

$$\frac{\Delta^2 u_x}{h^2} = \frac{d}{dx}\left(\frac{\Delta u_x}{h}\right) = \frac{d^2 u_x}{dx^2}; \text{ \&c.}$$

Hence, preparing the formula as follows,

$$u_{x+nh} = u_x + nh\frac{\Delta u_x}{h} + \frac{nh(nh-h)}{1\,.\,2}\frac{\Delta^2 u_x}{h^2} + \text{\&c.},$$

and taking the limit of both sides by supposing h to be infinitely diminished and n infinitely increased, their product always remaining equal to a finite magnitude t, we get

$$u_{x+t} = u_x + t\frac{du_x}{dx} + \frac{t^2}{1\,.\,2}\frac{d^2 u_x}{dx^2} + \text{\&c.}$$

The Differential Calculus is a particular case of that of Finite Differences; and the above investigation is introduced to show how, from results in Finite Differences obtained with an indeterminate increment for the principal variable, we may pass to the corresponding results in the Differential Calculus.

The theorems of Arts. 18 and 23 have been proved by an inductive process; they may also be established by the theory of Generating Functions, the principles of which we shall now proceed to explain; as it is a theory which, for its generality and power, especially merits our attention.

Generating Functions.

26. Let $\phi(t)$ be a function of t, susceptible of the development

$$\phi(t) = \ldots + u_{-1}t^{-1} + u_0 + u_1 t + \ldots$$
$$+ u_{x-1}t^{x-1} + u_x t^x + u_{x+1}t^{x+1} + \ldots;$$

then u_x may evidently represent any function of x whatever, if we regard this equation as the definition of $\phi(t)$. The

function $\phi(t)$ consequently by its development generates the coefficients $u_0, u_1, \ldots u_x$ annexed to their proper powers of t, and is therefore called the Generating Function of u_x, and is denoted by Gu_x, so that

$$\phi(t) = Gu_x.$$

Thus since

$$\log(1-t)^{-1} = t + \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 + \ldots + \frac{1}{x}t^x + \ldots$$

$$\log(1-t)^{-1} = G\frac{1}{x}.$$

Similarly, since

$$t(1-t)^{-2} = t + 2t^2 + 3t^3 + \ldots + xt^x + \ldots$$

$$t(1-t)^{-2} = Gx.$$

27. To determine the generating functions of u_{x+n} and $\Delta^n u_x$ from that of u_x.

Let $\phi(t) = Gu_x$,

then $\phi(t) = \quad \ldots + u_n t^n + u_{n+1}t^{n+1} + \ldots + u_{x+n}t^{x+n} + \ldots\ldots(1)$;

$$\therefore\ t^{-n}\phi(t) = \ldots + u_n + u_{n+1}t + \ldots + u_{x+n}t^x + \ldots$$

$$\therefore\ t^{-n}\phi(t) = Gu_{x+n}, \quad \text{or } t^{-n}Gu_x = Gu_{x+n}.$$

Again, $t^n\phi(t) = \ldots + u_{-n} + u_{-n+1}t + \ldots + u_{x-n}t^x + \ldots$

$$\therefore\ t^n\phi(t) = Gu_{x-n}, \quad \text{or } t^n Gu_x = Gu_{x-n}.$$

Hence it follows that the generating function of

$$\Delta u_x, \quad \text{or } u_{x+1} - u_x, \quad \text{is } \left(\frac{1}{t} - 1\right)\phi(t),$$

for this function being developed will produce the difference of two series whose general terms are respectively $u_{x+1}t^x$ and $u_x t^x$;

$$\therefore\ G(\Delta u_x) = \left(\frac{1}{t} - 1\right) Gu_x.$$

Similarly,

$$G(\Delta^2 u_x) = \left(\frac{1}{t} - 1\right) G(\Delta u_x) = \left(\frac{1}{t} - 1\right)^2 Gu_x,$$

$$\text{and } G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n Gu_x.$$

Also

$$G(\Delta^n u_{x-n}) = \left(\frac{1}{t} - 1\right)^n G u_{x-n} = \left(\frac{1}{t} - 1\right)^n t^n G u_x = (1-t)^n G u_x.$$

28. If instead of multiplying $\phi(t)$ by a power of t, we multiply it by another function of t, $\psi(t)$ similar to $\phi(t)$, that is, capable of being developed in a series of integral powers of t positive or negative, so that

$$\psi(t) = \ldots + p_{-1}t^{-1} + p_0 + p_1 t + p_2 t^2 + \ldots + p_x t^x + \ldots\ldots(2),$$

then in the product we find for the coefficient of t^x,

$$\ldots + p_{-1} u_{x+1} + p_0 u_x + p_1 u_{x-1} + p_2 u_{x-2} + \ldots$$

which may be replaced (Art. 17) by

$$\ldots + p_{-1} D u_x + p_0 u_x + p_1 D^{-1} u_x + p_2 D^{-2} u_x + \ldots;$$

and this, separating the symbols, may be written

$$(\ldots + p_{-1} D + p_0 + p_1 D^{-1} + p_2 D^{-2} + \ldots)\, u_x,$$

or $\psi\left(\frac{1}{D}\right) u_x$, (each term of the development of $\psi\left(\frac{1}{D}\right)$ being understood to be prefixed to u_x); which shews that $\psi(t) \times \phi(t)$ is the generating function of $\psi\left(\frac{1}{D}\right) u_x$; in other words,

$$\psi(t) \times G u_x = G\left\{\psi\left(\frac{1}{D}\right) u_x\right\}.$$

And if in (2) we replace t by t^{-1} it may be shewn in exactly the same way, that $\psi\left(\frac{1}{t}\right) \times \phi(t)$ is the generating function of $\psi(D) u_x$, that is

$$\psi\left(\frac{1}{t}\right) \times G u_x = G\{\psi(D) u_x\}.$$

Suppose for example, $\psi(t)$ to assume successively the forms t^{-n}, $(t^{-1} - 1)^n$; then, as before, we get

$$t^{-n} \times G u_x = G(D^n u_x) = G(u_{x+n}),$$

$$(t^{-1} - 1)^n \times G u_x = G\{(D-1)^n u_x\} = G(\Delta^n u_x).$$

29. To investigate the expression for $\Delta^n u_x$ in terms of u_{x+n}, u_{x+n-1}, &c., by Generating Functions.

$$G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n Gu_x$$

$$= t^{-n} Gu_x - nt^{-n+1} Gu_x + \frac{n(n-1)}{1.2} t^{-n+2} Gu_x - \&c.$$

$$= Gu_{x+n} - nGu_{x+n-1} + \frac{n(n-1)}{1.2} Gu_{x+n-2} - \&c.$$

$$= G\left(u_{x+n} - nu_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \&c.\right)$$

$$\therefore \Delta^n u_x = u_{x+n} - nu_{x+n-1} + \frac{n(n-1)}{1.2} u_{x+n-2} - \&c.$$

for, the generating functions of both being the same, the coefficients of t^x in the developments of those functions must be identical, however those developments have been effected.

30. To investigate the expression for u_{x+n} in terms of u_x and its first n differences, by Generating Functions.

$$Gu_{x+n} = t^{-n} Gu_x$$

$$= \left\{1 + \left(\frac{1}{t} - 1\right)\right\}^n Gu_x$$

$$= Gu_x + n\left(\frac{1}{t} - 1\right) Gu_x + \frac{n(n-1)}{1.2}\left(\frac{1}{t} - 1\right)^2 Gu_x + \&c.$$

$$= Gu_x + nG(\Delta u_x) + \frac{n(n-1)}{1.2} G(\Delta^2 u_x) + \&c.$$

$$= G\left(u_x + n\Delta u_x + \frac{n(n-1)}{1.2} \Delta^2 u_x + \&c.\right);$$

$$\therefore u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{1.2} \Delta^2 u_x + \&c.$$

31. It is obvious that by transforming the expressions

$$\left(\frac{1}{t} - 1\right)^n Gu_x, \quad \text{and } t^{-n} Gu_x,$$

in different ways, we may obtain various other expressions for $\Delta^n u_x$ and u_{x+n} besides the above.

Thus to express $\Delta^n u_x$ in terms of $\Delta^n u_{x-n}$, $\Delta^{n+1} u_{x-n-1}$, &c., we have

$$G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n Gu_x = \frac{(1-t)^n Gu_x}{\{1-(1-t)\}^n}$$

$$= (1-t)^n Gu_x + n(1-t)^{n+1} Gu_x$$

$$+ \frac{n(n+1)}{1.2}(1-t)^{n+2} Gu_x + \text{\&c.}$$

$$= G(\Delta^n u_{x-n}) + nG(\Delta^{n+1} u_{x-n-1}) + \frac{n(n+1)}{1.2} G(\Delta^{n+2} u_{x-n-2}) + \text{\&c.}$$

$$= G\left\{\Delta^n u_{x-n} + n\Delta^{n+1} u_{x-n-1} + \frac{n(n+1)}{1.2}\Delta^{n+2} u_{x-n-2} + \text{\&c.}\right\};$$

$$\therefore\ \Delta^n u_x = \Delta^n u_{x-n} + n\Delta^{n+1} u_{x-n-1} + \frac{n(n+1)}{1.2}\Delta^{n+2} u_{x-n-2} + \text{\&c.}$$

32. Again, to express u_{x+n} in terms of u_x, Δu_{x-r}, $\Delta^2 u_{x-2r}$, &c., we must transform t^{-n} into a series of powers of $t^r\left(\frac{1}{t} - 1\right)$, that is, we must develop t^{-n} in powers of x from the equation

$$\frac{1}{t} = 1 + x\left(\frac{1}{t}\right)^r,$$

which may be done by Lagrange's Theorem; and we find (see Herschel's Examples)

$$u_{x+n} = u_x + n\Delta u_{x-r} + \frac{n(n+2r-1)}{1.2}\Delta^2 u_{x-2r}$$

$$+ \frac{n(n+3r-1)(n+3r-2)}{1.2.3}\Delta^3 u_{x-3r} + \text{\&c.}$$

This method is obviously not confined to the function $u_{x+1} - u_x$; it is equally applicable to any other combination of the successive values u_x, u_{x+1}, u_{x+2}, &c., of the first degree. If we

take Δu_x to mean $au_{x+2} + bu_{x+1} + cu_x$, then the generating functions of Δu_x and $\Delta^n u_x$ will evidently be

$$\left(\frac{a}{t^2} + \frac{b}{t} + c\right) Gu_x, \qquad \left(\frac{a}{t^2} + \frac{b}{t} + c\right)^n Gu_x,$$

and the expression for $\Delta^n u_x$ in terms of u_x and its successive values, might be obtained as in the preceding case.

Separation of the Symbols of Operation from those of Quantity.

33. We have seen (Arts. 18 and 23) in the formulæ

$$D^n u_x = (1 + \Delta)^n u_x, \quad \Delta^n u_x = (D - 1)^n u_x,$$

instances of the method which consists in separating the symbols of operation from those of quantity; the use of which is not confined to simple cases like those just noticed, but may be extended with remarkable effect to a great variety of investigations connected with this subject.

By symbols of operation are meant certain characteristic letters placed before any functions, to denote that certain operations have been performed on them: thus D placed before a function of any variable x, denotes the operation of changing in it x into $x + h$; and Δ placed before the same function, implies that two different values of the variable have been substituted in it and the results subtracted from one another. By symbols of quantity are meant the subjects of the operations just mentioned; that is, letters taken to represent numbers, or algebraical expressions; and it must be remarked that these latter may be also regarded as symbols of operation. For if a be a number, then a denotes that the unit employed in the investigation, whatever it be, is to be added to itself n times; and a^2 or $a \, . \, a$ denotes that the same operation has been performed on a that was performed on unity; whenever therefore the two kinds of symbols occur together in the same formula, as in $(Da - 1)^n$, a must be regarded as a symbol of operation.

34.* The expressions $(1 + \Delta)^n$, $(D - 1)^n$ must be taken as abbreviated forms for their developments; and when prefixed to the function u_x, each term of these developments is understood to

be applied separately to that function. And, in general, if $F(\Delta)$ be a function of Δ capable of being developed in a series of powers of Δ such as

$$A\Delta^{\alpha} + B\Delta^{\beta} + \&c.; \quad \text{then for } A\Delta^{\alpha}u_x + B\Delta^{\beta}u_x + \&c.$$

the expression $F(\Delta)\, u_x$ is used as an abbreviation; and the same notation is applicable to other characteristic symbols, such as

$$\frac{d}{dx}, \ \int dx, \ \Sigma, \ \delta_x,$$

that in their combinations are subject to the same laws as algebraical quantities.

Also if we replace $\Delta^{\alpha}u_x$, $\Delta^{\beta}u_x$, &c. by their equivalents, we get

$$F(\Delta)\, u_x = A\,(D-1)^{\alpha}u_x + B\,(D-1)^{\beta}u_x + \&c.$$

$$= \{A\,(D-1)^{\alpha} + B\,(D-1)^{\beta} + \&c.\}\, u_x = F(D-1)\, u_x,$$

which shews that in any formula the symbols Δ and $D-1$ may be interchanged.

Hence also the successive performance of two or more series of operations represented by $F(\Delta)$, $F'(\Delta)$, upon the same function u_x, is equivalent to the performance of that series of operations denoted by their product. The method of separation of symbols is in every case capable of a strict inductive proof, and does not rest merely upon accidental analogies; and it deserves great notice on account of the facility with which it enables us to conduct many intricate processes. But as the generalizations which it offers, may present some difficulties to the student, we shall continue, as we have done hitherto, to obtain several of the principal results by an elementary process; before investigating them by the method of separation of symbols, or pointing out how they arise from that method.

We shall now give instances of the application of the method of separation of symbols, to obtain several important results.

34. To find the n^{th} difference of the product of two functions.

We have seen that $\Delta\,(u_x v_x) = \Delta u_x v_{x+1} + u_x \Delta v_x$

$$= \Delta u_x D v_x + u_x \Delta v_x = (\Delta D' + \Delta')\, u_x v_x,$$

separating the symbols, and supposing in the second member Δ to affect u_x only, and Δ' and D' to affect v_x only,

$$\text{then } \Delta^2(u_xv_x) = (\Delta D' + \Delta')\,\Delta\,(u_xv_x) = (\Delta D' + \Delta')^2 u_xv_x;$$

and, generally, $\Delta^n (u_xv_x) = (\Delta D' + \Delta')^n u_xv_x$

$$= (\Delta^n D'^n + n\Delta^{n-1}\Delta' D'^{n-1} + \frac{n(n-1)}{1.2}\Delta^{n-2}\Delta'^2 D'^{n-2} + \&c.)\, u_xv_x$$

$$= \Delta^n u_x \,.\, v_{x+n} + n\Delta^{n-1}u_x \,.\, \Delta v_{x+n-1} + \frac{n(n-1)}{1.2}\Delta^{n-2}u_x\Delta^2 v_{x+n-2} + \&c.,$$

which may be also proved inductively by shewing, as in Art. 13, that the coefficients of the developments of $\Delta^n(u_xv_x)$ and $(1+z)^n$, which are identical when $n=1$, undergo the same changes in passing from n to $n+1$.

If the series be reversed or, which is the same thing, if we develop the formula $\Delta^n (u_xv_x) = (\Delta' + \Delta D')^n u_xv_x$, we get

$$\Delta^n (u_xv_x) = u_x\Delta^n v_x + n\Delta u_x \Delta^{n-1} v_{x+1} + \frac{n(n-1)}{1.2}\Delta^2 u_x \Delta^{n-2} v_{x+2} + \&c.$$

Since $\Delta(u_xv_x) = u_{x+1}v_{x+1} - u_xv_x = Du_xDv_x - u_xv_x = (DD'-1)\,u_xv_x$ where D affects u_x only, and D' affects v_x only, we obtain another development of $\Delta^n(u_xv_x)$, viz.

$$\Delta^n(u_xv_x) = (DD'-1)^n u_xv_x$$

$$= (D^nD'^n - nD^{n-1}D'^{n-1} + \frac{n(n-1)}{1.\ 2}D^{n-2}D'^{n-2} - \&c.)\, u_xv_x$$

$$= u_{x+n}v_{x+n} - nu_{x+n-1}v_{x+n-1} + \frac{n(n-1)}{1.2}u_{x+n-2}v_{x+n-2} - \&c.$$

In the formula $\Delta^n(u_xv_x) = (DD'-1)^n u_xv_x$ suppose $v_x = a^x$, then $D'a^x = a^{x+1}$, so that the operation upon a^x denoted by D' is multiplying it by a,

$$\therefore\ \ \Delta^n(u_xa^x) = (Da-1)^n u_xa^x = a^x\,(Da-1)^n u_x.$$

Also $u_{x+n}\Delta^n v_x$

$= (\Delta' D)^n v_xu_x = (\Delta' D + \Delta - \Delta)^n v_xu_x$

$= \{(\Delta' D + \Delta)^n - n\,(\Delta' D + \Delta)^{n-1}\Delta + \&c.\}\ v_xu_x$

$= \Delta^n (v_xu_x) - n\Delta^{n-1}(v_x\Delta u_x) + \frac{1}{2}n(n-1)\,\Delta^{n-2}(v_x\Delta^2 u_x) - \dots \pm v_x\Delta^n u_x,$

a formula by which $u_{x+n}\Delta^n v_x$ is expressed by a series of differences with constant coefficients.

In the above instance we use an accent not to imply that the operations denoted by Δ, D, are altered at all, but merely that Δ', D', affect v_x only, whilst Δ, D affect u_x only. The above result is sometimes written,

$$\Delta^n (u_x v_x) = (DD' - 1)^n u_x v_x = \{(1+\Delta)(1+\Delta') - 1\}^n u_x v_x,$$

replacing D, D' by their equivalents; or, if there are more functions w_x, z_x, &c. and we use Δ'', Δ''', &c. to imply that in the second member these symbols only affect w_x, z_x, &c. respectively; and similarly for D, D', D'', &c.; we have

$$\Delta^n (u_x v_x w_x z_x \ldots) = (DD'D'' \ldots - 1)^n u_x v_x w_x \ldots$$

$$= \{(1+\Delta)(1+\Delta')(1+\Delta'')(1+\Delta''')\ldots - 1\}^n u_x v_x w_x z_x \ldots$$

35. To shew that $\Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x$, in which the symbols of operation are separated from those of quantity.

By Taylor's Theorem, we have

$$u_{x+n} = u_x + n\frac{du_x}{dx} + \frac{n^2}{1\,.\,2}\frac{d^2u_x}{dx^2} + \&\text{c.},$$

and separating the symbols of operation from those of quantity, we get

$$u_{x+n} = \left\{1 + n\frac{d}{dx} + \frac{n^2}{1\,.\,2}\left(\frac{d}{dx}\right)^2 + \frac{n^3}{1\,.\,2\,.\,3}\left(\frac{d}{dx}\right)^3 + \&\text{c.}\right\} u_x = e^{n\frac{d}{dx}} u_x.$$

Similarly, $u_{x+n-1} = e^{(n-1)\frac{d}{dx}} u_x$, &c. $u_{x+1} = e^{\frac{d}{dx}} u_x$;

$$\therefore\ \Delta^n u_x = e^{n\frac{d}{dx}} u_x - ne^{(n-1)\frac{d}{dx}} u_x + \frac{n(n-1)}{1\,.\,2} e^{(n-2)\frac{d}{dx}} u_x - \&\text{c.};$$

or, again separating the symbols of operation from those of quantity,

$$\Delta^n u_x = \left\{e^{n\frac{d}{dx}} - ne^{(n-1)\frac{d}{dx}} + \frac{n(n-1)}{1\,.\,2} e^{(n-2)\frac{d}{dx}} - \&\text{c.}\right\} u_x = \left(e^{\frac{d}{dx}} - 1\right)^n u_x,$$

a celebrated theorem first given by Lagrange.

36. To find a general expression for the n^{th} difference of a function in terms of its differential coefficients.

The development of the second member of the equation

$$\Delta^n u_x = \left(e^{\frac{d}{dx}} - 1\right)^n u_x$$

will consist of a series of terms of the form

$$\{A_0 + A_1 \frac{d}{dx} + \ldots + A_m \left(\frac{d}{dx}\right)^m + \ldots\} u_x,$$

$$\text{or } A_0 u_x + A_1 \frac{du_x}{dx} + \ldots + A_m \frac{d^m u_x}{dx^m} + \ldots,$$

and A_m is evidently the coefficient of t^m in the expansion of $(e^t - 1)^n$.

$$\text{Now } (e^t - 1)^n = e^{nt} - ne^{(n-1)t} + \frac{n(n-1)}{1.2} e^{(n-2)t} - \&\text{c.},$$

and the coefficients of t^m in the developments of e^{nt}, $e^{(n-1)t}$, &c. are respectively

$$\frac{n^m}{\lfloor m}, \quad \frac{(n-1)^m}{\lfloor m}, \quad \&\text{c.};$$

$$\therefore \; A_m = \frac{1}{\lfloor m} \{n^m - n(n-1)^m + \frac{n(n-1)}{1.2}(n-2)^m - \&\text{c.}\} = \frac{\Delta^n 0^m}{\lfloor m},$$

from Art. 20.

Now so long as $m < n$, this vanishes; and when $n = m$, $\Delta^n 0^m = \lfloor n$;

$$\therefore \; \Delta^n u_x = \frac{d^n u_x}{dx^n} + \frac{\Delta^n 0^{n+1}}{\lfloor n+1} \frac{d^{n+1} u_x}{dx^{n+1}} + \frac{\Delta^n 0^{n+2}}{\lfloor n+2} \frac{d^{n+2} u_x}{dx^{n+2}} + \&\text{c.}$$

Ex. Let $u_x = x^m$, then

$$\Delta^n (x^m) = m(m-1) \ldots (m-n+1)\, x^{m-n}$$

$$+ \frac{\Delta^n 0^{n+1}}{\lfloor n+1} \cdot m(m-1) \ldots (m-n)\, x^{m-n-1} + \&\text{c.} + \Delta^n 0^m.$$

37. If in Lagrange's Theorem for $\Delta^n u_x$, $n = 1$, we have $\Delta u_x = (e^{\frac{d}{dx}} - 1) u_x$, or $\Delta = e^{\frac{d}{dx}} - 1$, the meaning of which is, that the operation denoted by Δ is equivalent to the series of opera-

tions denoted by $e^{\frac{d}{dx}} - 1$. And generally, the series of operations denoted by $f(\Delta)$ is equivalent to that denoted by $f(e^{\frac{d}{dx}} - 1)$. For let $f(\Delta)$ be developed in a series of the form

$$f(\Delta) = A\Delta^{\alpha} + B\Delta^{\beta} + C\Delta^{\gamma} + \&c.,$$

then $f(\Delta)\, u_x = A\Delta^{\alpha} u_x + B\Delta^{\beta} u_x + C\Delta^{\gamma} u_x + \&c.$

$$= A\,(e^{\frac{d}{dx}} - 1)^{\alpha} u_x + B\,(e^{\frac{d}{dx}} - 1)^{\beta} u_x + C\,(e^{\frac{d}{dx}} - 1)^{\gamma} u_x + \&c.$$

or, separating the symbols of operation form those of quantity,

$$f(\Delta)\, u_x = \{A\,(e^{\frac{d}{dx}} - 1)^{\alpha} + B(e^{\frac{d}{dx}} - 1)^{\beta} + C\,(e^{\frac{d}{dx}} - 1)^{\gamma} + \&c.\}\; u_x$$

$$= f(e^{\frac{d}{dx}} - 1)\; u_x.$$

Thus, suppose $f(\Delta) = (1 + \Delta)^n$,

$$\text{then } f(e^{\frac{d}{dx}} - 1) = (1 + e^{\frac{d}{dx}} - 1)^n = e^{n\frac{d}{dx}};$$

therefore, annexing a function u_x for the symbols to operate upon,

$$(1 + \Delta)^n u_x = e^{n\frac{d}{dx}} u_x = u_{x+n},$$

as already proved.

Similarly, making $n = 1$ in the formula $u_{x+n} = e^{n\frac{d}{dx}} u_x$ (Art. 35), we find $u_{x+1} = e^{\frac{d}{dx}} u_x$, or $Du_x = e^{\frac{d}{dx}} u_x$; therefore $D = e^{\frac{d}{dx}}$, the meaning of which is that the operation denoted by D is equivalent to the series of operations denoted by $e^{\frac{d}{dx}}$; and, generally, the series of operations denoted by $f(D)$ is equivalent to that denoted by $f(e^{\frac{d}{dx}})$, which is expressed by the formula

$$f(D)\, u_x = f(e^{\frac{d}{dx}})\; u_x.$$

38. To express the n^{th} differential coefficient of any function by its differences.

Suppose $f(D) = (\log D)^n$, then $f(e^{\frac{d}{dx}}) = (\log e^{\frac{d}{dx}})^n = \left(\frac{d}{dx}\right)^n$;

$$\therefore\; \frac{d^n u_x}{dx^n} = (\log D)^n u_x = \{\log\,(1 + \Delta)\}^n u_x.$$

39. To find the general term of the expansion of $f(e^t)$ in a series ascending by powers of t.

Writing $f(e^t)$ in the form $f\{1+(e^t-1)\}$, and expanding by Taylor's theorem, we find

$$f(e^t)=f(1)+f'(1)(e^t-1)+\frac{1}{1.2}f''(1)(e^t-1)^2+\dots$$
$$+\frac{1}{\lfloor n}f^{(n)}(1)(e^t-1)^n+\dots$$

Then taking, as in Art. 36, the coefficient of t^m in each term of the second member, and observing that in $f(1)$ it may be represented by $\frac{f(1).0^m}{\lfloor m}$, this quantity being $f(1)$ when $m=0$, and zero in all other cases; and that in

$$\frac{1}{\lfloor n}f^{(n)}(1)(e^t-1)^n \text{ it is } \frac{1}{\lfloor n}f^{(n)}(1)\frac{\Delta^n 0^m}{\lfloor m},$$

we have for the coefficient of t^m in the expansion of $f(e^t)$, the value

$$A_m=\frac{1}{\lfloor m}\{f(1)\,0^m+f'(1)\,\Delta 0^m+\frac{1}{1.2}f''(1)\,\Delta^2 0^m+\dots\}$$
$$=\frac{1}{\lfloor m}f(1+\Delta)\,0^m,$$

a remarkable theorem first given by Herschel; for the applications of which, see his Collection of Examples. Hence in the development of e^{e^t} the coefficient of t^m is

$$\frac{1}{\lfloor m}e^{1+\Delta}0^m=\frac{e}{\lfloor m}\left(0^m+\Delta 0^m+\frac{1}{1.2}\Delta^2 0^m+\dots+\frac{\Delta^m 0^m}{\lfloor m}\right);$$

and in the series for $\frac{1}{e^t+1}$ the coefficient of t^m is

$$\frac{1}{\lfloor m}\frac{1}{2+\Delta}0^m=\frac{1}{\lfloor m}\left(\frac{0^m}{2}-\frac{\Delta 0^m}{2^2}+\frac{\Delta^2 0^m}{2^3}-\dots\pm\frac{\Delta^m 0^m}{2^{m+1}}\right);$$

the advantage of using the differences of zero being that any series of them necessarily terminates at $\Delta^m 0^m$.

SECTION II.

INVERSE METHOD OF DIFFERENCES.

Integration of Explicit Functions.

40. THE Inverse Method of Differences has for its object to determine the primitive function from its given difference; or from given relations between it and its differences. We shall begin with the simplest case,

$$\Delta u_x = f(x),$$

in which it is required to determine a function whose difference is given explicitly in terms of the principal variable.

41. Since Δu_x is the difference of $u_x + C$, as well as of u_x, it will be necessary, in passing from the given difference Δu_x to the primitive function, to annex an arbitrary constant C, in order to give the result all the generality of which it is capable. Also C may be a function of x as well as an arbitrary constant, provided its value remains unaltered whilst x changes to $x+1$. For if C_x denote such a function of x that $C_{x+1} = C_x$, or $\Delta C_x = 0$, we shall have

$$\Delta (u_x + C_x) = \Delta u_x.$$

It is evident that $C_x = \phi(2\lambda\pi x)$ has the property in question, ϕ denoting any trigonometrical function, sine, cosine, &c., and λ any integer. We shall see further on the importance of this remark.

42. The symbol Σ is used to denote the operation by which we pass from the difference Δu_x to the primitive function; so that

$$\Sigma (\Delta u_x) = u_x + \text{constant};$$

hence Σ and Δ denote operations the reverse of each other.

Also, as the same function admits of successive differences, so a function may be integrated any number of times; the second integral of u_x, or $\Sigma(\Sigma u_x)$, is written $\Sigma^2 u_x$, and the n^{th} integral $\Sigma^n u_x$.

If in the formula $\Sigma(\Delta u_x) = u_x$, we suppose the symbols of operation to be separated from those of quantity, we find $\Sigma\Delta u_x = u_x$; but on the same supposition $\Delta^{-1}\Delta u_x = \Delta^0 u_x = u_x$; so that Σ produces exactly the same effect as Δ^{-1}. Similarly Σ^n may be shewn to be identical with Δ^{-n}; so that in any formula a negative power of Δ may be always replaced by the same positive power of Σ, and *vice versâ.*

We now proceed to deduce the integrals of various expressions; chiefly, by reversing the processes given in Section I. for finding the differences of functions.

43. It is evident that $\Sigma(u_x + v_x + w_x) = \Sigma u_x + \Sigma v_x + \Sigma w_x$; for if we take the difference of both sides, we get the same result, viz. $u_x + v_x + w_x$. And in the same manner it appears that $\Sigma(au_x) = a\Sigma u_x$, and $\Sigma 0 = C$.

44. To find the integral of any rational integral function.

Since the difference of a rational integral function is a function of the same kind one dimension lower, it follows that the integral of a function of that description is a similar function one dimension higher; hence, to find the integral of

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots + p_{n-1} x + p_n,$$

we may assume it equal to

$$ax^{n+1} + bx^n + \dots + kx + l;$$

then upon taking the difference of each side, and equating the coefficients of like powers of x, there will arise $n+1$ simple equations to determine the $n+1$ quantities $a, b, c, \dots k$; the last term l will remain indeterminate, being in fact the arbitrary constant which must be added to make the integral complete.

Ex. To find $\Sigma(x^4 + 1)$.

Assume $\Sigma(x^4 + 1) = ax^5 + bx^4 + cx^3 + dx^2 + ex$;

$$\therefore x^4 + 1 = a(5x^4 + 10x^3 + 10x^2 + 5x + 1) + b(4x^3 + 6x^2 + 4x + 1)$$
$$+ c(3x^2 + 3x + 1) + d(2x + 1) + e;$$

$$\therefore\ 1 = 5a,\ \ 0 = 10a + 4b,\ \ 0 = 10a + 6b + 3c,\ \ 0 = 5a + 4b + 3c + 2d,$$

$$1 = a + b + c + d + e.$$

$$\therefore\ a = \frac{1}{5},\quad b = -\frac{1}{2},\quad c = \frac{1}{3},\quad d = 0,\quad e = \frac{29}{30},$$

$$\therefore\ \Sigma\,(x^4 + 1) = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} + \frac{29}{30}x + C.$$

45. To find the integral of the product of consecutive terms of an arithmetic progression, we must annex one more factor at the beginning, and divide by the number of factors so increased and by the common difference.

For let $u_x = a + bx$, then we have seen (Art. 11) that

$$\Delta u_x u_{x+1} \ldots u_{x+n} = u_{x+1} u_{x+2} \ldots u_{x+n} \,.\, (n+1)\, b,$$

therefore, taking the integrals of both sides, and writing $x - 1$ for x, we get

$$\Sigma u_x u_{x+1} \ldots u_{x+n-1} = \frac{u_{x-1} u_x \ldots u_{x+n-1}}{(n+1)\, b} + C,$$

which proves the rule stated above.

Ex. $\Sigma \left(2x + \frac{1}{2}\right)\left(2x + \frac{5}{2}\right)\left(2x + \frac{9}{2}\right)$

$$= \frac{1}{8}\left(2x - \frac{3}{2}\right)\left(2x + \frac{1}{2}\right)\left(2x + \frac{5}{2}\right)\left(2x + \frac{9}{2}\right) + C.$$

Each factor of an expression capable of being integrated by this rule, must be derivable from the preceding factor by changing x into $x + 1$.

If one or more factors be deficient in a factorial of this kind, it may be resolved into others which are complete, as in the following instance;

$$(2x + 1)(2x + 5)(2x + 7) = (2x + 3 - 2)(2x + 5)(2x + 7)$$

$$= (2x + 3)(2x + 5)(2x + 7) - 2\,(2x + 5)(2x + 7).$$

46. A rational integral function may often be resolved into factorials of the above form, and in this way its integral more conveniently found, than by the method of Art. 44.

Ex. 1. $x^3 + x^2 = x^2(x+1) = (x-1+1)\,x\,(x+1)$

$$= (x-1)\,x\,(x+1) + x\,(x+1)\,;$$

$$\therefore\ \Sigma(x^3+x^2) = \frac{(x-2)(x-1)\,x\,(x+1)}{4} + \frac{(x-1)\,x\,(x+1)}{3} + C.$$

And in general, any quantity of the form

$$ax^n + bx^{n-1} + cx^{n-2} + \&c.$$

may be resolved into factorials, by the method of indeterminate coefficients; thus, if we assume

$$ax^2 + bx + c = A\,(x+1)(x+2) + B\,(x+1) + C,$$

making $x = -1$, we get $a - b + c = C$;

$$\therefore\ a\,(x^2-1) + b\,(x+1) = A\,(x+1)(x+2) + B\,(x+1),$$

$$\text{or}\ \ a\,(x-1) + b = A\,(x+2) + B\,;$$

$$\text{make}\ \ x = -2,\quad \therefore\ -3a + b = B\,;$$

$$\therefore\ a\,(x-1) + 3a = A\,(x+2),\quad \therefore\ A = a.$$

In practice, however, it is generally easier to resolve a function by inspection, as in Ex. 1, than by this method, which is theoretically certain.

47. To find the integral of a fraction whose denominator is the product of consecutive terms of an arithmetic progression, and numerator constant, we must efface the last factor, divide by the number of factors remaining and by the common difference, and prefix a negative sign.

For let $u_x = a + bx$, then we have seen (Art. 12) that

$$\Delta\frac{c}{u_x u_{x+1}\ldots u_{x+n-1}} = -\frac{nbc}{u_x u_{x+1}\ldots u_{x+n}}\,;$$

therefore taking the integral of both sides,

$$\Sigma \frac{c}{u_x u_{x+1} \dots u_{x+n}} = -\frac{c}{nbu_x u_{x+1} \dots u_{x+n-1}} + C,$$

which proves the rule just stated.

48. If the proposed fraction, instead of having its numerator constant, be

$$\frac{Ax^{n-2} + Bx^{n-3} + \dots + Kx + L}{u_x u_{x+1} \dots u_{x+n-1}},$$

(the degree of the numerator being at least lower by two units than that of the denominator,) we must reduce the numerator to a series of terms each of which is the product of consecutive factors reckoning from the beginning of the denominator; that is, assume

$$Ax^{n-2} + Bx^{n-3} + \dots + Kx + L = A' + B'u_x + C'u_x u_{x+1} + \dots$$
$$+ K'u_x u_{x+1} \dots u_{x+n-3},$$

then, developing the second member, and equating coefficients of like powers of x, we obtain $n-1$ equations for determining A', B', C', ... K'; and the fraction resolves itself into the following, each of which is integrable,

$$\frac{A'}{u_x \dots u_{x+n-1}} + \frac{B'}{u_{x+1} \dots u_{x+n-1}} + \&c. + \frac{K'}{u_{x+n-2} \cdot u_{x+n-1}}.$$

If the degree of the numerator were the same as that of the denominator, or only lower by one unit than that of the denominator, we should arrive at a term $\frac{L'}{u_{x+n-1}}$, of which we are able to find the integral, only approximately.

Hence also, if any of the factors of the denominator of the fraction in Art. 47 be wanting, they may be supplied by introducing them into the numerator and denominator at the same time; and then the resulting fraction may be treated as in the present Article.

Ex. 1. $\dfrac{1}{x^2-4}=\dfrac{(x-1)\,x\,(x+1)}{(x-2)\dots(x+2)}$

$$=\frac{1}{(x+1)(x+2)}+\frac{3}{x(x+1)(x+2)}+\frac{6}{(x-1)x(x+1)(x+2)}$$

$$+\frac{6}{(x-2)(x-1)x(x+1)(x+2)},$$

which is got by assuming

$$(x-1)x(x+1)=a(x-2)(x-1)x+b(x-2)(x-1)+c(x-2)+d,$$

and making $x=2, 1, 0$, successively; taking care to reject the factor common to both sides, after each substitution.

Ex. 2. $\Sigma\dfrac{1}{4x^2-9}=-\dfrac{1}{6}\,\dfrac{12x^2-12x-1}{(2x-3)(4x^2-1)}.$

49. The fraction in the preceding Art. may be also integrated when the denominator is the product of any number of factors x, $x+mh$, $x+nh$, $x+rh$, &c., m, n, r, &c., denoting whole numbers, and h the increment of x.

For by taking the difference of both members, we perceive the truth of the result (which, although expressed in a series the number of whose terms is variable, is often useful)

$$\Sigma\frac{M}{x(x+mh)}=-\frac{M}{mh}\left\{\frac{1}{x}+\frac{1}{x+h}+\dots+\frac{1}{x+(m-1)h}\right\}\dots(1).$$

But $\dfrac{x+a}{x(x+mh)(x+nh)}=\dfrac{A}{x(x+mh)}+\dfrac{B}{x(x+nh)}\dots\dots(2),$

where $A+B=1,\quad (An+Bm)\,h=a;$

so that this fraction is integrable by formula (1). Next multiply both sides of (2) by $\dfrac{x+b}{x+rh}$; then the second member of the result can be resolved by (2) into fractions having a constant numerator, and the product of two simple factors for denominator, and is therefore integrable by (1); and so on to any number of factors, the dimension of the numerator being always less by at least two units than that of the denominator.

50. To find the integrals of a^x, and $\log v_x$.

We have seen (Art. 14) that $\Delta a^x = (a-1)\,a^x$;

$$\therefore\ \Sigma a^x = \frac{a^x}{a-1} + C. \quad \text{Also } \Sigma^n a^x = \frac{a^x}{(a-1)^n},$$

suppressing the part introduced by the constants, which would be a rational integral function of the $(n-1)^{\text{th}}$ degree, and might be represented by $\Sigma^n 0$, since $\Sigma 0 = C$, $\Sigma^2 0 = Cx + C'$, &c. The formula seems to fail when $a=1$, as it gives infinity for the value of Σa^x in that case, instead of x; but if we give the constant C the form $C' - \dfrac{1}{a-1}$, then

$$\Sigma a^x = \frac{a^x - 1}{a-1} + C';$$

now let $a = 1 + h$, where h is very small, then

$$\Sigma a^x = \frac{1 + hx + \frac{1}{2}x\,(x-1)\,h^2 + \&\text{c.} - 1}{h} = x + \tfrac{1}{2}x\,(x-1)\,h + \&\text{c.};$$

therefore, when $a=1$, $\Sigma a^x = x + C'$.

Next to find the integral of $\log v_x$.

If $u_x = \log(v_1 v_2 v_3 \ldots v_{x-1})$, then

$$\Delta u_x = \log(v_1 v_2 \ldots v_x) - \log(v_1 v_2 \ldots v_{x-1}) = \log v_x;$$

$$\therefore\ \Sigma \log v_x = u_x + \log C = \log(C \,.\, v_1 v_2 \ldots v_{x-1}) = \log C P v_{x-1},$$

using Pv_x to denote the product of all the successive values of the function v_x, from some fixed term v_1 (or more generally v_n, n being independent of x) to v_x inclusive.

51. To find the integrals of $\cos x\theta$, $\sin x\theta$.

$$\text{Since } \Delta \cos x\theta = -2 \sin \tfrac{1}{2}\theta \sin (x + \tfrac{1}{2})\,\theta,$$

$$\therefore\ \Delta \cos (x - \tfrac{1}{2})\,\theta = -2 \sin \tfrac{1}{2}\theta \sin x\theta;$$

$$\therefore\ \Sigma \sin x\theta = -\frac{\cos (x - \frac{1}{2})\,\theta}{2 \sin \frac{1}{2}\theta} + C.$$

Also since $\Delta^n \sin(x\theta + \alpha) = (2\sin\frac{1}{2}\theta)^n \sin\{x\theta + \alpha + \frac{1}{2}n(\pi + \theta)\}$, (Art. 16), integrating both sides n times, and replacing α by $\alpha - \frac{1}{2}n(\pi + \theta)$, we get

$$\Sigma^n \sin(x\theta + \alpha) = \frac{\sin\{x\theta + \alpha - \frac{1}{2}n(\pi + \theta)\}}{(2\sin\frac{1}{2}\theta)^n};$$

the same result as if in the value of $\Delta^n \sin(x\theta + \alpha)$ we had changed the sign of n; as we should expect, Δ^{-n} being equivalent to Σ^n.

Again, $\Delta \sin x\theta = 2\sin\frac{1}{2}\theta \cos(x + \frac{1}{2})\theta$,

$$\therefore\ \Delta \sin(x - \tfrac{1}{2})\theta = 2\sin\tfrac{1}{2}\theta \cos x\theta;$$

$$\therefore\ \Sigma \cos x\theta = \frac{\sin(x - \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} + C;$$

$$\text{and } \Sigma^n \cos(x\theta + \alpha) = \frac{\cos\{x\theta + \alpha - \frac{1}{2}n(\pi + \theta)\}}{(2\sin\frac{1}{2}\theta)^n},$$

changing the sign of n in the value of $\Delta^n \cos(x\theta + \alpha)$ in Art. 16.

52. The preceding expressions may also be integrated by substituting for them their exponential values; as in the following instance.

$$\Sigma a^x \cos x\theta = \tfrac{1}{2}\Sigma\left(a^x e^{x\theta\sqrt{-1}} + a^x e^{-x\theta\sqrt{-1}}\right)$$

$$= \frac{1}{2}\frac{a^x e^{x\theta\sqrt{-1}}}{ae^{\theta\sqrt{-1}} - 1} + \frac{1}{2}\frac{a^x e^{-x\theta\sqrt{-1}}}{ae^{-\theta\sqrt{-1}} - 1}$$

$$= \frac{a^x}{2}\,\frac{ae^{(x-1)\theta\sqrt{-1}} + ae^{-(x-1)\theta\sqrt{-1}} - e^{x\theta\sqrt{-1}} - e^{-x\theta\sqrt{-1}}}{a^2 - a\left(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right) + 1}$$

$$= a^x\,\frac{a\cos(x-1)\theta - \cos x\theta}{a^2 - 2a\cos\theta + 1} + C.$$

Hence, putting $a^2 - 2a\cos\theta + 1 = c$, and denoting

$a^x \cos x\theta$ by u_x, we have $c\Sigma u_x = a^2 u_{x-1} - u_x$,

$$c^2\Sigma^2 u_x = a^4 u_{x-2} - 2a^2 u_{x-1} + u_x;$$

and generally

$$c^n\Sigma^n u_x = a^{2n} u_{x-n} - na^{2n-2} u_{x-n+1} + \frac{n(n-1)}{1.2}a^{2n-4} u_{x-n+2} - \&c. \pm u_x.$$

This result may be obtained immediately by separation of symbols; for we have

$$\Sigma u_x = \frac{1}{c}\,(a^2 u_{x-1} - u_x) = \frac{1}{c}\,(a^2 D^{-1} - 1)\, u_x;$$

$$\text{consequently } \Sigma^n u_x = \frac{1}{c^n}\,(a^2 D^{-1} - 1)^n u_x;$$

and this when developed produces the preceding result.

Exactly the same formulæ hold for $u_x = a^x \sin x\theta$.

In the same manner the integrals of $a^x (\sin x\theta)^m$, $a^x (\cos x\theta)^m$ may be obtained, when the powers of the sine and cosine of $x\theta$ have been replaced by the sines and cosines of multiples of $x\theta$.

53. To find the integral of $\dfrac{1}{\cos x\theta \cos (x+1)\theta}$.

Since $\Delta \tan x\theta = \dfrac{\sin\theta}{\cos x\theta \cos (x+1)\theta}$, (Art. 16*.)

$$\text{we have } \Sigma \frac{1}{\cos x\theta \cos (x+1)\theta} = \frac{\tan x\theta}{\sin\theta} + C.$$

54. To find the integral of $\tan^{-1} \dfrac{1}{p + qx + rx^2}$.

Since $\Delta \tan^{-1}(a + bx) = \tan^{-1} \dfrac{b}{1 + (a+bx)(a+bx+b)}$: (Art. 16*.)

$$\text{we may assume } \Sigma \tan^{-1} \frac{1}{p + qx + rx^2} = \tan^{-1}(a + bx),$$

and take the difference of both sides; then if the proposed function is capable of being integrated, the indeterminate coefficients a and b will become known. Also by differencing $n^x \tan^{-1}(\theta n^{-x})$ we may find an expression of which it is the integral, when $n = 2, 3$, &c.

55. The integral of $a^x u_x u_{x+1} \dots u_{x+n-1}$, where $u_x = pa^x + q$, may be determined by assuming it equal to the same expression (only with another factor at the beginning instead of a^x) multiplied by an indeterminate coefficient, and taking the difference of both sides; for

$$\begin{aligned}\Delta u_{x-1} u_x \dots u_{x+n-1} &= u_x u_{x+1} \dots u_{x+n-1} (u_{x+n} - u_{x-1}) \\ &= a^x u_x u_{x+1} \dots u_{x+n-1} \cdot p\,(a^n - a^{-1});\end{aligned}$$

$$\therefore\ \Sigma a^x u_x \ldots u_{x+n-1} = \frac{a}{p\,(a^{n+1}-1)} \cdot u_{x-1} u_x \ldots u_{x+n-1}.$$

Similarly, to find $\Sigma \dfrac{a^x}{u_x u_{x+1} \ldots u_{x+n}}$, for the assumption we must efface the last factor in the denominator, and write instead of a^x an indeterminate coefficient; for

$$\Delta \frac{1}{u_x u_{x+1} \ldots u_{x+n-1}} = -\frac{u_{x+n}-u_x}{u_x u_{x+1} \ldots u_{x+n}} = -\frac{pa^x\,(a^n-1)}{u_x u_{x+1} \ldots u_{x+n}};$$

$$\therefore\ \Sigma \frac{a^x}{u_x u_{x+1} \ldots u_{x+n}} = -\frac{1}{p\,(a^n-1)} \cdot \frac{1}{u_x u_{x+1} \ldots u_{x+n-1}};$$

56. In like manner, if $u_x = a + bx$, expressions of the forms

$$\frac{(p+qx)\,t^x}{u_x u_{x+1} \ldots u_{x+n-1}},\qquad \frac{(p+qx+rx^2)\,t^x}{u_x u_{x+1} \ldots u_{x+n-1}},$$

can sometimes be integrated, by assuming their integrals equal to expressions of the same form, except that the last factor in the denominator is effaced, and the polynomial in the numerator is replaced by another one dimension lower with indeterminate coefficients. It is of course only when a certain equation of condition between the quantities a, b, p, q, t is satisfied, that this method succeeds.

Ex. 1. Let $\Sigma \dfrac{x+1}{(2x-1)\,(2x+1)\,3^x} = \dfrac{A\,(\frac{1}{3})^x}{2x-1}$;

$$\therefore\ \frac{x+1}{(2x-1)\,(2x+1)\,3^x} = A\,(\tfrac{1}{3})^{x+1}\left(\frac{1}{2x+1} - \frac{3}{2x-1}\right)$$

$$= A\,(\tfrac{1}{3})^{x+1}\,\frac{-4\,(x+1)}{(2x-1)\,(2x+1)};$$

$$\therefore\ A = -\tfrac{3}{4},\ \text{and}\ \Sigma u_x = C - \tfrac{3}{4}\,\frac{1}{(2x-1)\,3^x}.$$

Ex. 2. Let $\Sigma \dfrac{x^2+6x+12}{x\,(x+1)\,(x+2)\,2^x} = \dfrac{A+Bx}{x\,(x+1)\,2^x}$,

then $A = -6$, $B = -2$, and $\Sigma u_x = C - \dfrac{x+3}{x\,(x+1)\,2^{x-1}}$.

57. Since $\Delta(u_x v_x) = u_x \Delta v_x + v_{x+1}\Delta u_x$, we have

$$\Sigma(u_x \Delta v_x) = u_x v_x - \Sigma(v_{x+1}\Delta u_x),$$

the formula for integration by parts, corresponding to the formula

$$\int dx u \frac{dv}{dx} = uv - \int dx v \frac{du}{dx}.$$

Change v_x into Σv_x, and consequently Δv_x into v_x,

$$\text{then} \quad \Sigma(u_x v_x) = u_x \Sigma v_x - \Sigma(\Delta u_x \Sigma v_{x+1});$$

hence, by successive substitutions, we get

$$\Sigma(\Delta u_x \Sigma v_{x+1}) = \Delta u_x \Sigma^2 v_{x+1} - \Sigma(\Delta^2 u_x \Sigma^2 v_{x+2}),$$

$$\Sigma(\Delta^2 u_x \Sigma^2 v_{x+2}) = \Delta^2 u_x \Sigma^3 v_{x+2} - \Sigma(\Delta^3 u_x \Sigma^3 v_{x+3}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Sigma(\Delta^n u_x \Sigma^n v_{x+n}) = \Delta^n u_x \Sigma^{n+1} v_{x+n} - \Sigma(\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1});$$

$$\therefore \Sigma(u_x v_x) = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2}$$
$$- \&c. \pm \Delta^n u_x \Sigma^{n+1} v_{x+n} \mp \Sigma(\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1}).$$

58. The above formula, replacing in the second member Σ by its equivalent Δ^{-1}, may be written

$$\Sigma(u_x v_x) = u_x \Delta^{-1} v_x - \Delta u_x . \Delta^{-2}(Dv_x) + \Delta^2 u_x . \Delta^{-3}(D^2 v_x) - \&c.$$
$$= (\Delta' + \Delta D')^{-1} u_x v_x,$$

if we restrict Δ to affect u_x only and Δ' and D' to affect v_x only; hence $\Sigma^2(u_x v_x) = (\Delta' + \Delta D')^{-2} u_x v_x$; and generally

$$\Sigma^n(u_x v_x) = (\Delta' + \Delta D')^{-n} u_x v_x \quad \text{.........} \quad (1)$$

$$= \left(\Delta'^{-n} - n\Delta'^{-n-1}\Delta D' + \frac{n(n+1)}{1.2}\Delta'^{-n-2}\Delta^2 D'^2 - \&c.\right) u_x v_x,$$

$$\text{or} \quad \Sigma^n(u_x v_x) = u_x \Sigma^n v_x - n\Delta u_x \Sigma^{n+1} v_{x+1}$$
$$+ \frac{n(n+1)}{1.2}\Delta^2 u_x \Sigma^{n+2} v_{x+2} - \&c. \ldots (2),$$

which may be also proved inductively, by shewing that the coefficients of the developments of $\Sigma^n(u_x v_x)$ and $(1+z)^{-n}$, which

are identical when $n=1$, undergo the same changes in passing from n to $n-1$. This appears from differencing both sides of equation (2). Both the formula (1) for $\Sigma^n(u_x v_x)$, and its development (2), result immediately, as we should expect, from the expressions for $\Delta^n(u_x v_x)$ in Art. 34, by changing the sign of n.

If in the formula $\Sigma^n(u_x v_x) = (\Delta' + \Delta D')^{-n} u_x v_x$ where Δ affects u_x only, and Δ', D' affect v_x only, we suppose $v_x = a^x$; then

$$\Delta' v_x = (a-1)\,a^x, \quad D' v_x = a \,.\, a^x;$$

and the formula becomes

$$\Sigma^n(u_x a^x) = (a-1+\Delta a)^{-n} u_x a^x = a^x (Da-1)^{-n} u_x,$$

which agrees with the result in Art. 34 when n has its sign changed.

If in this last formula we change a into $\frac{1}{a}$, we get

$$(D-a)^{-n} u_x = a^{x-n} \Sigma^n (u_x a^{-x}),$$

$$\text{and} \quad (D-a)^{-1} u_x = a^{x-1} \Sigma (u_x a^{-x}).$$

59. The above formula for $\Sigma^n(u_x v_x)$ always enables us to find the integrals of functions made up of two factors, one of which leads to zero, as the value of one of its successive differences, and the other admits of successive integrations. Suppose, for example, that u_x is a rational integral function of the n^{th} degree, and that $v_x = a^x$; then

$$\Sigma v_x = \frac{a^x}{a-1}, \quad \Sigma^2 v_{x+1} = \frac{a^{x+1}}{(a-1)^2}, \text{ \&c.}; \quad \Delta^n u_x = \text{const}, \quad \Delta^{n+1} u_x = 0;$$

$$\therefore \Sigma\,(u_x a^x) = \frac{u^x a^x}{a-1} - \frac{\Delta u_x a^{x+1}}{(a-1)^2} + \frac{\Delta^2 u_x a^{x+2}}{(a-1)^3} - \dots \pm \frac{\Delta^n u_x a^{x+n}}{(a-1)^{n+1}} + C.$$

Again, suppose u_x to be a rational integral function of the n^{th} degree, and $v_x = \cos x\theta$; then taking the value of $\Sigma^n v_x$ from Art. 51,

$$\Sigma\,(u_x \cos x\theta) = \frac{u_x \cos\{x\theta - \frac{1}{2}(\pi+\theta)\}}{2\sin\frac{1}{2}\theta}$$

$$-\Delta u_x \frac{\cos\{(x+1)\theta - (\pi+\theta)\}}{(2\sin\frac{1}{2}\theta)^2} + \Delta^2 u_x \frac{\cos\{(x+2)\theta - \frac{3}{2}(\pi+\theta)\}}{(2\sin\frac{1}{2}\theta)^3} - \text{\&c.},$$

the series terminating with $\Delta^n u_x$. Similarly, if $v_x = a^x \cos x\theta$; or if $v_x = a^x \cos^m x\theta . \sin^n x\theta$, since the product $\cos^m x\theta . \sin^n x\theta$ may be replaced by simple dimensions of sines and cosines of multiples of $x\theta$; and it will be noticed that the fraction of Art. 48 may be brought under this case.

60. Since the performance of the operation Σ upon any series of terms $A\Delta^m u_x + B\Delta^n u_x + \ldots$, reduces it to

$$A\Delta^{m-1} u_x + B\Delta^{n-1} u_x + \ldots;$$

it appears that prefixing Σ to $(A\Delta^m + B\Delta^n + \ldots.)\, u_x$ has the same effect as prefixing Δ^{-1}; in other words, Σ is equivalent to Δ^{-1}. And in like manner, since integrating $\Delta^m u_x$ n times, reduces it to $\Delta^{m-n} u_x$, Σ^n must be equivalent to Δ^{-n}.

The same reasoning is applicable to the symbols $(\int dx)^n$, $\left(\frac{d}{dx}\right)^n$; whenever therefore, in separating the symbols of operation from those of quantity, as in the expression $F(\Delta)u_x$, $f\left(\frac{d}{dx}\right) u_x$, terms containing negative powers of Δ and $\frac{d}{dx}$ occur, they must be understood to be replaced by the corresponding positive powers of Σ and $\int dx$. This being premised, we proceed to investigate a general series for Σu_x; preparatory to which the following propositions must be proved.

61. To determine the generating functions of

$$\Sigma^n u_x, \quad \frac{d^n u_x}{dx^n}, \quad \int^n dx^n u_x,$$

from that of u_x.

By virtue of the relations

$$G(\Delta u_x) = \left(\frac{1}{t} - 1\right) G u_x,$$

$$G(\Delta^n u_x) = \left(\frac{1}{t} - 1\right)^n G u_x, \quad \ldots\ldots\ldots\ldots\ldots (1)$$

we have $\left(\frac{1}{t} - 1\right) G(\Sigma u^x) = G(\Delta\Sigma u_x) = G u_x$;

$$\therefore\ G(\Sigma u_x) = \left(\frac{1}{t} - 1\right)^{-1} Gu_x,$$

$$\text{and} \quad G(\Sigma^n u_x) = \left(\frac{1}{t} - 1\right)^{-n} Gu_x,$$

the same result as if we had changed the sign of n in (1) and replaced Δ^{-n} by Σ^n.

62. Again, since

$$Gu_x = \ldots + u_x t^x + \ldots + u_{x+h} t^{x+h} + \ldots;$$

$$\therefore\ \ldots + (u_{x+h} - u_x)\, t^x + \ldots = \left(\frac{1}{t^h} - 1\right) Gu_x$$

$$= \left\{1 - h \log t + \frac{h^2}{1\,.\,2} (\log t)^2 - \&\text{c.} - 1\right\} Gu_x;$$

therefore, dividing both sides by h and then making $h = 0$, which we are at liberty to do since h is indeterminate,

$$\ldots + \left\{\frac{u_{x+h} - u_x}{h}\right\}_{h=0} t^x + \ldots = -\log t Gu_x = \log \frac{1}{t}\,.\, Gu_x,$$

$$\text{or} \quad \ldots + \frac{du_x}{dx}\,.\, t^x + \ldots = \log \frac{1}{t}\,.\, Gu_x;$$

$$\therefore\ G\left(\frac{du_x}{dx}\right) = \log \frac{1}{t}\,.\, Gu_x,$$

$$\text{and} \quad G\left(\frac{d^n u_x}{dx^n}\right) = \left(\log \frac{1}{t}\right)^n Gu_x \ldots\ldots\ldots\ldots\ldots (1).$$

Again, $\log \frac{1}{t}\, G(\int dx u_x) = G\left(\frac{d}{dx} \int dx u_x\right) = Gu_x;$

$$\therefore\ G(\int dx u_x) = \left(\log \frac{1}{t}\right)^{-1} Gu_x, \text{ and } G(\int^n dx^n u_x) = \left(\log \frac{1}{t}\right)^{-n} Gu_x;$$

the same result as if we had changed the sign of n in (1) and replaced $\left(\frac{d}{dx}\right)^{-n}$ by $(\int dx)^n$.

These values for $G\left(\frac{d^n u_x}{dx^n}\right)$, $G(\int^n dx^n u_x)$, may be immediately obtained from Art. 28; for if in the formula

$$G\{\psi(D)\, u_x\} = \psi\left(\frac{1}{t}\right).\, Gu_x,$$

we suppose $\psi\left(\frac{1}{t}\right) = \left(\log\frac{1}{t}\right)^n$, then

$$\psi(D) = (\log D)^n = (\log e^{\frac{d}{dx}})^n = \left(\frac{d}{dx}\right)^n,$$

$$\therefore\ G\left(\frac{d^n u_x}{dx^n}\right) = \left(\log\frac{1}{t}\right)^n . Gu_x;$$

and changing the sign of n, and replacing $\left(\frac{d}{dx}\right)^{-n}$ by $(\int dx)^n$,

$$G(\int^n dx^n u_x) = \left(\log\frac{1}{t}\right)^{-n} . Gu_x.$$

63. To investigate a general series for Σu_x, involving only $\int dx u_x$, u_x, and the differential coefficients of u_x.

$$G(\Sigma u_x) = \left(\frac{1}{t} - 1\right)^{-1} Gu_x = (e^{\log\frac{1}{t}} - 1)^{-1} Gu_x$$

$$= \left(\log\frac{1}{t}\right)^{-1} . \frac{\log\frac{1}{t}}{e^{\log\frac{1}{t}} - 1} . Gu_x$$

$$= \left(\log\frac{1}{t}\right)^{-1} \left\{1 - \tfrac{1}{2}\log\frac{1}{t} + \frac{B_1}{1.2}\left(\log\frac{1}{t}\right)^2 - \&c.\right.$$

$$\left. + (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}\left(\log\frac{1}{t}\right)^{2n} + \ldots\right\} Gu_x,$$

assuming, as will be proved in the next Art., that $\frac{v}{e^v - 1}$ can be expanded in a series of the same form as that within brackets, and denoting by

$$\frac{B_1}{1.2},\ \frac{-B_3}{1.2.3.4},\ \&c.,\ (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n},$$

the coefficients of $v^2, v^4, \ldots v^{2n}$ in that expansion. Hence

$$G(\Sigma u_x) = G(\int dx u_x) - \tfrac{1}{2}Gu_x + \frac{B_1}{1.2}G\left(\frac{du_x}{dx}\right) - \&c.$$

$$= G\left\{\int dx u_x - \tfrac{1}{2}u_x + \frac{B_1}{1\,.\,2}\frac{du_x}{dx} - \frac{B_3}{\lfloor 4}\frac{d^3u_x}{dx^3} + \&c.\right\};$$

$$\therefore\ \Sigma u_x = \int dx u_x - \tfrac{1}{2}u_x + \frac{B_1}{1\,.\,2}\frac{du_x}{dx} - \frac{B_3}{\lfloor 4}\frac{d^3u_x}{dx^3} + \ldots$$

$$+ (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}\frac{d^{2n-1}u_x}{dx^{2n-1}} + \ldots$$

Ex. 1. $\Sigma x^m = \dfrac{x^{m+1}}{m+1} - \tfrac{1}{2}x^m + \tfrac{1}{2}B_1 m x^{m-1}$

$$- \tfrac{1}{4}B_3\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}x^{m-3} + \&c. + C.$$

Hence $\Sigma x^4 = \dfrac{x^5}{5} - \dfrac{x^4}{2} + \dfrac{x^3}{3} - \dfrac{x}{30} + C;$

$$\Sigma x^5 = \frac{x^6}{6} - \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12} + C.$$

Ex. 2. $\Sigma\dfrac{1}{a+bx} = \dfrac{1}{b}\log(a+bx) - \dfrac{1}{2(a+bx)}$

$$- \frac{B_1 b}{2(a+bx)^2} + \frac{B_3 b^3}{4(a+bx)^4} - \&c. + C.$$

Similarly $\Sigma^n u_x$ may be expressed in terms of the integrals and differential coefficients of u_x by making n negative in the formula of Art. 35, and expanding $e^{\frac{d}{dx}}$, which gives

$$\Sigma^n u_x = \left(\frac{d}{dx} + \frac{1}{1\,.\,2}\frac{d^2}{dx^2} + \frac{1}{1\,.\,2\,.\,3}\frac{d^3}{dx^3} + \ldots\right)^{-n} u_x.$$

Numbers of Bernouilli.

64. The numbers B_1, B_3, ... which are required in the general value of Σu_x in the preceding Art., are called the Numbers of Bernouilli, and are of great importance in the theory of Series. They are defined by the equation

$$\frac{t}{e^t - 1} = 1 - \tfrac{1}{2}t + \frac{B_1}{1\,.\,2}t^2 - \frac{B_3}{\lfloor 4}t^4 + \ldots + (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}t^{2n} \pm \&c.,$$

and their values may be computed in the following manner.

$$\text{If } \phi(t) = \frac{t}{e^t - 1}, \text{ then } \phi(-t) = \frac{-t}{e^{-t} - 1} = \frac{te^t}{e^t - 1},$$

$$\therefore \phi(t) - \phi(-t) = -t;$$

which shews that the only term in the development of $\phi(t)$ which involves an odd power of t is $-\frac{1}{2}t$; for if any higher odd power entered, it would occur in $\phi(t) - \phi(-t)$ with its coefficient doubled; we may assume therefore for $\phi(t)$ the form of development given above, viz.

$$\frac{t}{e^t - 1} = 1 - \tfrac{1}{2}t + \frac{B_1}{1.2}t^2 - \frac{B_3}{\lfloor 4}t^4 + \ldots + (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}t^{2n} \pm \&\text{c. (1)}.$$

$$\text{Now } \frac{t}{e^t - 1} = \frac{\log\{1 + (e^t - 1)\}}{e^t - 1} = 1 - \tfrac{1}{2}(e^t - 1) + \tfrac{1}{3}(e^t - 1)^2 - \ldots$$

$$+ \frac{1}{2n+1}(e^t - 1)^{2n} - \&\text{c.},$$

and if A_{2n} be the coefficient of t^{2n} in this development of the second member,

$$A_{2n} = -\tfrac{1}{2}\frac{1^{2n}}{\lfloor 2n} + \tfrac{1}{3}\cdot\frac{2^{2n} - 2 \cdot 1^{2n}}{\lfloor 2n} - \&\text{c.}$$

$$+ \frac{1}{2n+1}\cdot\frac{(2n)^{2n} - 2n(2n-1)^{2n} + \ldots}{\lfloor 2n} - \&\text{c.}$$

$$= \frac{1}{\lfloor 2n}\{-\tfrac{1}{2}\Delta 0^{2n} + \tfrac{1}{3}\cdot\Delta^2 0^{2n} - \&\text{c.} + \frac{1}{2n+1}\Delta^{2n}0^{2n}\};$$

all the terms after $\Delta^{2n}0^{2n}$ vanishing, since $\Delta^m 0^{2n}$ is zero when $m > 2n$. Hence

$$B_{2n-1} = (-1)^{n+1}\{-\tfrac{1}{2}\Delta 0^{2n} + \tfrac{1}{3}\Delta^2 0^{2n} - \&\text{c.} \ldots + \frac{1}{2n+1}\Delta^{2n}0^{2n}\}.$$

By this formula B_1, B_3, B_5, &c. may be readily computed, supposing the numbers comprised in the form $\Delta^m 0^{2n}$, or the differences of zero to be known; we find

$$B_1 = \frac{1}{6}, \quad B_3 = \frac{1}{30}, \quad B_5 = \frac{1}{42}, \quad B_7 = \frac{1}{30}, \quad B_9 = \frac{5}{66}, \ \&\text{c.}$$

values that may be easily verified by applying Maclaurin's Theorem to obtain the development (1).

65. Also, since

$$\Delta^m 0^{2n} = m^{2n} - m(m-1)^{2n} + \frac{m(m-1)}{1.2}(m-2)^{2n} - \&c. \text{ (Art. 20)}$$

we may, if we please, eliminate the differences of zero from the expression for B_{2n-1}; and we find

$$(-1)^{n+1} B_{2n-1} = -\tfrac{1}{2} + \tfrac{1}{3}(2^{2n}-2) - \tfrac{1}{4}(3^{2n} - 3.2^{2n} + 3) + \&c. + \frac{\lfloor 2n}{2n+1}.$$

Besides serving to express the general value of Σu_x, the numbers of Bernouilli have various other uses, of which we shall now give one or two of the most remarkable. Any function that can be put under the form $t \div (e^t - 1)$ may be expanded in terms involving those numbers.

66. To find the general terms of the expansions of cot θ and tan θ in powers of θ.

$$\cot\theta = \sqrt{-1}\left(1 + \frac{2}{e^{2\theta\sqrt{-1}} - 1}\right) = \sqrt{-1} + \frac{1}{\theta}\cdot\frac{2\theta\sqrt{-1}}{e^{2\theta\sqrt{-1}} - 1}.$$

Now the general term of the expansion of

$$\frac{2\theta\sqrt{-1}}{e^{2\theta\sqrt{-1}} - 1} \text{ is } (-1)^{n+1}\frac{B_{2n-1}}{\lfloor 2n}(2\theta\sqrt{-1})^{2n}, \text{ or } -\frac{2^{2n}B_{2n-1}}{\lfloor 2n}\theta^{2n};$$

$\therefore$ the general term of the expansion of cot θ is $-\dfrac{2^{2n}B_{2n-1}}{\lfloor 2n}\theta^{2n-1}$,

$$\text{and } \cot\theta = \frac{1}{\theta} - \frac{2^2}{1.2}B_1\theta - \frac{2^4 B_3}{1.2.3.4}\theta^3 - \&c.$$

Also since $\tan\theta = \cot\theta - 2\cot 2\theta$,

the general term of the series for tan θ is

$$-\frac{2^{2n}B_{2n-1}\theta^{2n-1}}{\lfloor 2n} + 2.\frac{2^{2n}B_{2n-1}(2\theta)^{2n-1}}{\lfloor 2n} = \frac{2^{2n}(2^{2n}-1)B_{2n-1}}{\lfloor 2n}\theta^{2n-1};$$

$$\therefore \tan\theta = \frac{4.3}{1.2}B_1\theta + \frac{16.15}{1.2.3.4}B_3\theta^3 + \frac{2^6(2^6-1)}{1.2\ldots 6}B_5\theta^5 + \&c.$$

Hence, by differentiating the above expressions for cot θ and tan θ, we may deduce the general terms of the expansions of

$\cot^2\theta$ and $\tan^2\theta$; and by integrating them, the general terms of the expansions of $\log\sin\theta$, $\log\cos\theta$.

67. To find S the sum of the infinite series

$$\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots$$

Since $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\dots$

changing θ into $\pi\theta$, we get

$$\sin\pi\theta=\pi\theta\left(1-\frac{\theta^2}{1^2}\right)\left(1-\frac{\theta^2}{2^2}\right)\left(1-\frac{\theta^2}{3^2}\right)\dots$$

therefore, differentiating the logarithm of each member,

$$\pi\cot\pi\theta=\frac{1}{\theta}-\frac{2\theta}{1^2}\left(1-\frac{\theta^2}{1^2}\right)^{-1}-\frac{2\theta}{2^2}\left(1-\frac{\theta^2}{2^2}\right)^{-1}-\frac{2\theta}{3^2}\left(1-\frac{\theta^2}{3^2}\right)^{-1}-\&c.$$

$$=\frac{1}{\theta}-2\theta\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)-2\theta^3\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right)-\&c.$$

$$-2\theta^{2n-1}\left(\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\right)-\dots$$

But the coefficient of θ^{2n-1} in the expansion of $\pi\cot\pi\theta$ is

$$-\frac{2^{2n}B_{2n-1}\pi^{2n}}{\lfloor 2n},$$

$$\therefore\ S=\frac{2^{2n-1}B_{2n-1}\pi^{2n}}{1.2.3\dots 2n}.$$

Hence $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6};\ \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90}.$

Calling S_{2n} the sum of this infinite series, we have

$$\frac{S_{2n}}{S_{2n+2}}=\frac{(2n+1)(2n+2)}{4\pi^2}\frac{B_{2n-1}}{B_{2n+1}}.$$

Now suppose n very great, then $\frac{B_{2n+1}}{B_{2n-1}} = \frac{n^2}{\pi^2}$, which proves the divergency of the series formed by the numbers of Bernouilli; these numbers increase very rapidly, beginning with B_{13}.

68. To find S the sum of the infinite series

$$\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \dots.$$

$$\text{Since } \cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\dots$$

changing θ into $\frac{\pi\theta}{2}$, we get $\cos\frac{\pi\theta}{2} = \left(1 - \frac{\theta^2}{1^2}\right)\left(1 - \frac{\theta^2}{3^2}\right)\left(1 - \frac{\theta^2}{5^2}\right)\dots$

therefore, differentiating the logarithm of each member,

$$\frac{\pi}{2}\tan\frac{\pi\theta}{2} = \frac{2\theta}{2}\left(1 - \frac{\theta^2}{1^2}\right)^{-1} + \frac{2\theta}{3^2}\left(1 - \frac{\theta^2}{3^2}\right)^{-1} + \frac{2\theta}{5^2}\left(1 - \frac{\theta^2}{5^2}\right)^{-1} + \&c.;$$

and equating the coefficients of θ^{2n-1} in each member, which are respectively,

$$\left(\frac{\pi}{2}\right)^{2n} B_{2n-1} \frac{2^{2n}(2^{2n}-1)}{\lfloor 2n}, \text{ and } 2S,$$

$$\text{we get } S = \tfrac{1}{2} B_{2n-1} \frac{\pi^{2n}(2^{2n}-1)}{1.2.3\dots 2n}.$$

$$\text{Hence } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \&c. = \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{3\pi^2}{1.2} = \frac{\pi^2}{8}.$$

69. Also $\frac{1}{1^{2n}} - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \dots$

$$= \frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \dots - \frac{1}{2^{2n}}\left(\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \dots\right)$$

$$= \frac{1}{2}\frac{2^{2n}-1}{\lfloor 2n}\pi^{2n} B_{2n-1} - \frac{1}{2^{2n}}\frac{2^{2n-1}\pi^{2n}}{\lfloor 2n} B_{2n-1}$$

$$= \frac{(2^{2n-1}-1)\pi^{2n} B_{2n-1}}{\lfloor 2n}.$$

Since $\log (1 + e^{-z}) = e^{-z} - \frac{1}{2} e^{-2z} + \frac{1}{3} e^{-3z} - \&c.$,

integrating this $2n-1$ times between the limits $z = 0$, $z = \infty$, we find for result the series just summed;

$$\therefore \ (\int dz_0^\infty)^{2n-1} \log (1 + e^{-z}) = \frac{(2^{2n-1} - 1)\, \pi^{2n} B_{2n-1}}{\lfloor 2n}.$$

70. To find an approximate value of

$$\Gamma (x + 1) = 1 \,.\, 2 \,.\, 3 \ldots x, \text{ when } x \text{ is very large.}$$

Making $u_x = \log x$ in the formula

$$\Sigma u_x = \int dx u_x - \tfrac{1}{2} u_x + \frac{B_1}{1 \,.\, 2} \frac{du_x}{dx} - \&c.$$

we find, (Art. 50) adding $\log x$ to both sides,

$$\log \{1 \,.\, 2 \,.\, 3 \ldots (x-1)\, x\} = C + x \log x - x - \tfrac{1}{2} \log x$$

$$+ \frac{B_1}{1 \,.\, 2} \frac{1}{x} - \frac{B_3}{3 \,.\, 4} \frac{1}{x^3} + \ldots + \log x$$

$$= C + (x + \tfrac{1}{2}) \log x - x + \log (1 + h),$$

$$\text{putting } \log (1 + h) = \frac{B_1}{1 \,.\, 2x} - \frac{B_3}{3 \,.\, 4x^3} + \frac{B_5}{5 \,.\, 6x^5} - \&c.,$$

so that h is a quantity continually approaching to zero as x increases.

Now to determine C, suppose x very large so that h may be neglected, and change x into $2x$, then

$$\log (1 \,.\, 2 \,.\, 3 \ldots 2x) = C + (2x + \tfrac{1}{2}) \log 2x - 2x$$

$$= C + (2x + \tfrac{1}{2}) (\log x + \log 2) - 2x,$$

$$\text{and } \log (2 \,.\, 4 \,.\, 6 \ldots 2x) = \log (2^x \,.\, 1 \,.\, 2 \,.\, 3 \ldots x)$$

$$= x \log 2 + C + (x + \tfrac{1}{2}) \log x - x;$$

$$\therefore \ \log \{1 \,.\, 3 \,.\, 5 \ldots (2x-1)\} = x \log x + (x + \tfrac{1}{2}) \log 2 - x;$$

$$\therefore \log \frac{2\,.\,4\,.\,6 \ldots 2x}{1\,.\,3\,.\,5 \ldots (2x-1)} = C + \tfrac{1}{2}\log x - \tfrac{1}{2}\log 2$$

$$= C + \tfrac{1}{2}\log 2x - \log 2;$$

$$\therefore\ 2C - 2\log 2 = 2\log \frac{2\,.\,4\,.\,6 \ldots (2x)}{1\,.\,3\,.\,5 \ldots (2x-1)} - \log 2x$$

$$= \log \frac{2\,.\,2\,.\,4\,.\,4\,.\,6\,.\,6 \ldots (2x-2)\,2x}{1\,.\,3\,.\,3\,.\,5\,.\,5\,.\,7 \ldots (2x-1)\,(2x-1)}$$

$$= \log \frac{\pi}{2}, \text{ by Wallis's Theorem,}$$

since x is indefinitely large;

$$\therefore\ C = \tfrac{1}{2}\log 2\pi;$$

$$\therefore \log (1\,.\,2\,.\,3 \ldots x) = \tfrac{1}{2}\log 2\pi + (x + \tfrac{1}{2})\log x - x + \log\,(1+h)$$

$$= \log\sqrt{2\pi x} + \log\left(\frac{x}{e}\right)^x + \log\,(1+h);$$

$$\therefore\ 1\,.\,2\,.\,3 \ldots x = \sqrt{2\pi x}\,.\left(\frac{x}{e}\right)^x.\,(1+h),$$

where h is to be calculated from the series

$$\log\,(1+h) = \frac{B_1}{1\,.\,2}\,.\,\frac{1}{x} - \frac{B_3}{3\,.\,4}\,.\,\frac{1}{x^3} + \frac{B_5}{5\,.\,6}\,.\,\frac{1}{x^5} - \&\text{c.},$$

and in general it will suffice to take the first term only, which gives

$$1\,.\,2\,.\,3 \ldots x = \sqrt{2\pi x}\,x^x e^{-x+\frac{1}{12x}};$$

or, more accurately, $1\,.\,2\,.\,3 \ldots x = \sqrt{2\pi x}\,x^x e^{-x+\frac{\mu}{12x}}$, where μ denotes a quantity lying between 0 and 1.

Obs. The preceding series, even for large values of x, becomes divergent after a certain number of terms; this will happen after n terms if

$$\frac{B_{2n+1}}{(2n+1)\,(2n+2)}\,.\,\frac{1}{x^{2n+1}} > \frac{B_{2n-1}}{(2n-1)\,2n}\,\frac{1}{x^{2n-1}},$$

$$\text{or}\ \frac{B_{2n+1}}{B_{2n-1}} > \frac{(2n+1)\,(2n+2)}{(2n-1)\,2n}x^2,$$

but the first member of this inequality never exceeds

$$\frac{(2n+1)(2n+2)}{4\pi^2}, \text{ (Art. 67.)}$$

$$\therefore\ (2n-1)\,2n > 4\pi^2x^2, \quad \text{or } n > \pi x.$$

It can, however, be proved that an approximate value of the series will be obtained by taking the aggregate of the convergent terms only, as will be seen in the next Article.

71. An approximate value of Σu_x will be obtained by taking the aggregate of the converging terms only, in the series for Σu_x, involving $\int dx u_x$, u_x, and the differential coefficients of u_x; and the error will be less than the last of the convergent, or the first of the divergent terms.

We have by Art. 63, (omitting the index of u_x in the second member,)

$$\Sigma u_x = C + \int dx u - \tfrac{1}{2}u + \frac{B_1}{1\,.\,2}\frac{du}{dx} - \frac{B_3}{\lfloor 4}\frac{d^3u}{dx^3} + \dots$$

$$+ (-1)^{n-1}\frac{B_{2n-1}}{\lfloor 2n}\frac{d^{2n-1}u}{dx^{2n-1}} + (-1)^n 2R_n, \qquad (1)$$

$$\text{where } 2R_n = \frac{B_{2n+1}}{\lfloor 2n+2}\frac{d^{2n+1}u}{dx^{2n+1}} - \frac{B_{2n+3}}{\lfloor 2n+4}\frac{d^{2n+3}u}{dx^{2n+3}} + \&c.;$$

or since in general, by Art. 67,

$$\frac{B_{2n-1}}{\lfloor 2n} = 2\left\{\frac{1}{(2\pi)^{2n}} + \frac{1}{(4\pi)^{2n}} + \frac{1}{(6\pi)^{2n}} + \&c.\right\};$$

$$R_n = \left\{\frac{1}{(2\pi)^{2n+2}} + \frac{1}{(4\pi)^{2n+2}} + \frac{1}{(6\pi)^{2n+2}} + \dots\right\}\frac{d^{2n+1}u}{dx^{2n+1}}$$

$$- \left\{\frac{1}{(2\pi)^{2n+4}} + \frac{1}{(4\pi)^{2n+4}} + \frac{1}{(6\pi)^{2n+4}} + \dots\right\}\frac{d^{2n+3}u}{dx^{2n+3}}$$

$$+ \left\{\frac{1}{(2\pi)^{2n+6}} + \frac{1}{(4\pi)^{2n+6}} + \frac{1}{(6\pi)^{2n+6}} + \dots\right\}\frac{d^{2n+5}u}{dx^{2n+5}}$$

$$- \&c.;$$

or, adding the terms vertically, and calling the resulting series v_1, v_2, &c.,

$$R_n = v_1 + v_2 + v_3 + \dots + v_m + \dots$$

$$\text{where } v_m = \frac{1}{(2m\pi)^{2n+2}}\frac{d^{2n+1}u}{dx^{2n+1}} - \frac{1}{(2m\pi)^{2n+4}}\frac{d^{2n+3}u}{dx^{2n+3}}$$

$$+ \frac{1}{(2m\pi)^{2n+6}}\frac{d^{2n+5}u}{dx^{2n+5}} - \&\text{c.};$$

$$\therefore \frac{d^2 v_m}{dx^2} = -(2m\pi)^2 \left\{ v_m - \frac{1}{(2m\pi)^{2n+2}}\frac{d^{2n+1}u}{dx^{2n+1}} \right\},$$

$$\text{or } \frac{d^2 v_m}{dx^2} + (2m\pi)^2 v_m = \frac{1}{(2m\pi)^{2n}}\frac{d^{2n+1}u}{dx^{2n+1}},$$

and integrating this by the method of parameters, as in Art. 72,

$$v_m = -\frac{1}{(2m\pi)^{2n+1}}\int dx \left(\sin 2m\pi x \,.\, \frac{d^{2n+1}u}{dx^{2n+1}}\right),$$

since, x being an integer, the term multiplied by $\sin 2m\pi x$ disappears, and $\cos 2m\pi x = 1$; and the arbitrary constant is unnecessary, being already introduced in equation (1);

$$\therefore R_n = -\int dx \left\{ \frac{\sin 2x\pi}{(2\pi)^{2n+1}} + \frac{\sin 4x\pi}{(4\pi)^{2n+1}} + \frac{\sin 6x\pi}{(6\pi)^{2n+1}} + \&\text{c.} \right\} \frac{d^{2n+1}u}{dx^{2n+1}};$$

therefore, numerically, $2R_n$ is less than

$$-2\int dx \left\{ \frac{1}{(2\pi)^{2n}} + \frac{1}{(4\pi)^{2n}} + \frac{1}{(6\pi)^{2n}} + \dots \right\} \frac{d^{2n+1}u}{dx^{2n+1}} < -\frac{B_{2n-1}}{\underline{|2n}}\frac{d^{2n}u}{dx^{2n}};$$

which last quantity lies between $\dfrac{B_{2n-1}}{\underline{|2n}}\dfrac{d^{2n-1}u}{dx^{2n-1}}$ and $\dfrac{B_{2n+1}}{\underline{|2n+2}}\dfrac{d^{2n+1}u}{dx^{2n+1}}$,

if these be the last of the convergent and first of the divergent terms respectively of the series for Σu_x. Consequently the sum of all the divergent terms in the series for Σu_x is less than the last of the convergent or the first of the divergent terms.

SECTION III.

EQUATIONS OF DIFFERENCES.

72. WE now come to the case in which the relation between the principal variable and any function of it is to be determined by means of an equation between x, u_x, and one or more of the successive values, or differences, of u_x; that is, from equations of the form

$$F(x, u_x, u_{x+1}, \dots u_{x+n}) = 0,$$

$$\text{or, } f(x, u_x, \Delta u_x, \dots \Delta^n u_x) = 0,$$

since, by the theorems of Arts. 18 and 23, these forms are convertible one into the other. An Equation of Differences is said to be of the n^{th} order when the successive value, or the difference, of the highest order which it involves is the n^{th}. An Equation of Differences of any order is said moreover to be of the first, second, &c. degree, when the successive value, or the Difference, which marks its order, is raised at most to the first, second, &c. power; or, when it involves a product of successive values or differences at most of two, three, &c. dimensions, it is said to be of the second, third, &c. degree.

73. The complete integral of an equation of differences of the n^{th} order will contain n arbitrary constants.

Let $\dots u_x, u_{x+h}, u_{x+2h}, \dots$ be a series of terms corresponding to the successive values $x, x+h, x+2h, \dots$; and let

$$F(x, u_x, a) = 0$$

be the equation by which the general term is determined as a function of x and a, or the equation of the series, a being an arbitrary constant. Since this equation must hold for all the succeeding terms, we shall have

$$F(x+h, u_{x+h}, a) = 0.$$

Eliminating a between these two equations, we get an equation between x, u_x, and u_{x+h}; or, substituting $u_x + \Delta u_x$, for u_{x+h}, an equation between x, u_x, and Δu_x, which is the equation of differences of the first order whose primitive equation is

$$F(x, u_x, a) = 0.$$

In like manner if the equation of the general term contained two arbitrary constants a and b, as

$$F(x, u_x, a, b) = 0,$$

we might eliminate a and b by means of the two succeeding equations,

$$F(x+h, u_{x+h}, a, b) = 0, \quad F(x+2h, u_{x+2h}, a, b) = 0,$$

and thus get an equation between x, u_x, u_{x+h}, u_{x+2h}; or, substituting

$$u_x + \Delta u_x \text{ for } u_{x+h}, \text{ and } u_x + 2\Delta u_x + \Delta^2 u_x \text{ for } u_{x+2h},$$

an equation between x, u_x, Δu_x, $\Delta^2 u_x$, without the constants a and b, which is an equation of differences of the second order, having for its complete primitive the equation

$$F(x, u_x, a, b) = 0.$$

Hence it appears that every equation of differences of the first order, or between two successive terms of a series, will introduce one arbitrary constant into the equation of the series; every equation of differences of the second order, or between three successive terms, will introduce two arbitrary constants into the equation of the series; and, generally, every equation of differences of the n^{th} order will introduce n arbitrary constants into the equation of the series.

Linear Equation of Differences of the First Order.

74. The general equation of the first order and degree is

$$u_{x+1} - A_x u_x = B_x,$$

A_x and B_x being functions of x. To integrate it, assume

$$u_x = v_x w_x, \quad \therefore \; v_{x+1}(w_x + \Delta w_x) - A_x v_x w_x = B_x;$$

and in order that this equation may resolve itself into two others each of which admits of being integrated, assume (as we are at liberty to do, having made only one supposition respecting v_x and w_x)

$$v_{x+1} w_x - A_x v_x w_x = 0,$$

$$\text{or, dividing by } v_x w_x, \; \frac{v_{x+1}}{v_x} = A_x.$$

$$\text{But } \Delta \log v_x = \log \frac{v_{x+1}}{v_x}, \quad \therefore \; \Delta \log v_x = \log A_x,$$

$$\therefore \; \log v_x = \Sigma \log A_x = \log PA_{x-1}, \quad \text{(Art. 50.)}$$

or $v_x = PA_{x-1}$, the constant being unnecessary.

The other part of the equation gives $v_{x+1} \Delta w_x = B_x$,

$$\therefore \; \Delta w_x = \frac{B_x}{PA_x}, \text{ or } w_x = \Sigma \left(\frac{B_x}{PA_x}\right) + C,$$

$$\text{and } u_x = PA_{x-1} \left\{\Sigma \left(\frac{B_x}{PA_x}\right) + C\right\}$$

the complete integral, involving one arbitrary constant.

Taking the difference of the result

$$\frac{u_x}{PA_{x-1}} = \Sigma \left(\frac{B_x}{PA_x}\right) + C, \text{ we get } \frac{u_{x+1} - A_x u_x}{PA_x} = \frac{B_x}{PA_x},$$

which shews that $\dfrac{1}{PA_x}$ is a factor which makes each side of the proposed equation integrable; and it is generally the most convenient way of integrating the equation to multiply it by this factor.

Ex. 1. $u_{x+1} - a u_x = x^2$.

Here $A_x = a$, $\quad PA_x = a^x$;

$$\therefore \frac{u_x}{a^{x-1}} = \Sigma\left(\frac{x^2}{a^x}\right) = \Sigma\,(x^2\alpha^x), \text{ putting } \frac{1}{a} = \alpha,$$

$$= \frac{x^2\alpha^x}{\alpha - 1} - \frac{(2x+1)\,\alpha^{x+1}}{(\alpha-1)^2} + \frac{2\alpha^{x+2}}{(\alpha-1)^3} + C; \text{ (Art. 59.)}$$

$$\therefore u_x = \frac{x^2}{1-a} - \frac{2x+1}{(1-a)^2} + \frac{2}{(1-a)^3} + Ca^{x-1}.$$

Ex. 2. $u_{x+1} = 2 \,.\, \dfrac{2x+1}{x+2}\, u_x.$

$$u_x = C \,.\, 2^x \frac{1 \,.\, 3 \,.\, 5 \dots (2x-1)}{1 \,.\, 2 \,.\, 3 \dots (x+1)}.$$

Ex. 3. $u_{x+1} - au_x = \cos x\theta.$

$$u_x = \frac{\cos (x-1)\,\theta - a\cos x\theta}{a^2 - 2a\cos\theta + 1} + Ca^x.$$

Ex. 4. Two vessels which hold a and b gallons respectively are filled, the one with proof spirit, the other with water; c gallons are taken from each and poured into the other; and this is repeated such a number of times as to make their contents of the same strength; find the number of times

$$x = \frac{\log \frac{1}{2}\left(1 - \frac{a}{b}\right)}{\log\left(1 - \frac{c}{a} - \frac{c}{b}\right)}.$$

Indirect Integrals of Equations of Differences.

75. Since the equation of differences of the first order,

$$f(x, u_x, \Delta u_x) = 0,$$

is formed by eliminating the constant a between the equations

$$u_x = F(x, a), \quad u_{x+h} = F(x+h, a),$$

it follows that we shall arrive at the same equation of differences, whether a be constant, or be a function of x such as a_x, provided it satisfies the condition

$$F(x+h, a_{x+h}) - F(x+h, a_x) = 0 \quad (1).$$

Now this equation is satisfied by $a_{x+h} = a_x$, which gives $\Delta a_x = 0$, and $a_x = a$, a constant, and leads to the ordinary or direct equation to the series $u_x = F(x, a)$.

Also the first member of (1) will be divisible by $a_{x+h} - a_x$, because a_x is a value of a_{x+h}, which satisfies equation (1); and if dimensions of a_{x+h} and a_x superior to the first are involved in it, the result of this division will be an equation involving a_{x+h} and a_x; i.e. an equation of differences of the first order with respect to a_x, the solution of which will give one or more values of a_x in terms of x and arbitrary constants; and these being substituted for a_x in the equation $u_x = F(x, a_x)$, will furnish equations of series, which are primitive equations of

$$f(x, u_x, \Delta u_x) = 0,$$

and each involves an arbitrary constant.

If equation (1) does not involve higher dimensions of a_x and a_{x+h} than the first, they will disappear from the result when it is divided by $a_{x+h} - a_x$. In this case a_x will have only one value, viz. $a_x = a$, and there will be only one equation of a series corresponding to the proposed equation of differences. The mode in which the indirect solutions just treated of are obtained, is analogous to that in which the singular solutions of differential equations are obtained; but whereas the latter can contain no arbitrary constant, indirect solutions of equations of differences may contain as many arbitrary constants as the complete integral itself from which they are deduced.

Ex. $u_x = x\Delta u_x + F(\Delta u_x)$.

Taking the Difference, we find

$$\Delta u_x = \Delta u_x + (x+1)\,\Delta^2 u_x + \Delta F(\Delta u_x),$$

$$\text{or } 0 = (x+1)\,\Delta^2 u_x + \Delta F(\Delta u_x),$$

which is evidently satisfied by $\Delta u_x = a$, a constant;

$$\therefore\ u_x = ax + a';$$

and substituting in the proposed equation

$$ax + a' = ax + F(a), \quad \therefore \ a' = F(a);$$

$$\therefore \ u_x = xa + F(a),$$

the complete integral, containing one arbitrary constant.

For the indirect solutions we shall have

$$u_x = xa_x + F(a_x),$$

a_x being determined from the equation

$$(x+1)\, a_{x+1} + F(a_{x+1}) - (x+1)\, a_x - F(a_x) = 0.$$

Suppose, for instance, that $F(a_x) = a^2_x$,

$$\therefore \ (x+1)\,(a_{x+1} - a_x) + a^2_{x+1} - a^2_x = 0,$$

or, rejecting the factor $a_{x+1} - a_x$,

$$a_{x+1} + a_x = -(x+1);$$

$$\therefore \ \frac{a_x}{(-1)^{x-1}} = -\Sigma\,(x+1)(-1)^x = \frac{2x+1}{4}(-1)^x + C,$$

$$\text{or } a_x = -\frac{2x+1}{4} + C(-1)^x;$$

$$\therefore \ u_x = xa_x + a^2_x = \frac{1-4x^2}{16} - \tfrac{1}{2}C(-1)^x + C'.$$

Linear Equations of Differences of all Orders.

The linear equation of Differences of the n^{th} order is

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \dots + p_n u_x = X,$$

all the coefficients being functions of x; the first step towards its integration is to establish the following theorem.

76. If there be n particular values v_x, w_x, ... z_x, which, when substituted for u_x, satisfy the equation

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \dots + p_n u_x = 0,$$

that has no term independent of u_x, its complete integral is

$$u_x = a_1 v_x + a_2 w_x + \dots + a_n z_x,$$

$a_1, a_2, \dots a_n$ being arbitrary constants.

For let this value be substituted in the expression

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \dots + p_n u_x,$$

and it becomes, (collecting the terms multiplied by the factors $a_1, a_2, \dots a_n$)

$$a_1 (v_{x+n} + p_1 v_{x+n-1} + \dots + p_n v_x) + a_2 (w_{x+n} + p_1 w_{x+n-1} + \dots + p_n w_x) + \dots$$
$$+ a_n (z_{x+n} + p_1 z_{x+n-1} + \dots + p_n z_x).$$

Now since $v_x, w_x, \dots z_x$, satisfy the proposed equation, each of the quantities included within brackets is equal to zero, therefore the whole is identically zero; consequently the assumed value of u_x satisfies the proposed equation, and it contains n arbitrary constants, therefore it is the complete integral of that equation.

77. To integrate the equation of differences,

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \dots + p_n u_x = q,$$

all the coefficients and q being constants.

Assume $u_x = v_x + k$; then by substitution we get

$$v_{x+n} + p_1 v_{x+n-1} + \dots + p_n v_x + k(1 + p_1 + \dots + p_n) - q = 0.$$

Let $k = \dfrac{q}{1 + p_1 + p_2 + \dots + p_n}$, then the equation becomes

$$v_{x+n} + p_1 v_{x+n-1} + p_2 v_{x+n-2} + \dots + p_n v_x = 0 \;\dots\dots\dots\; (1).$$

Let $v_x = a^x$, then $a^x (a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_n)$ is the value of the first member; now this will vanish if a be any root of the equation (called the auxiliary equation),

$$f(a) = a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_{n-1} a + p_n = 0.$$

Hence the n roots of this equation $a_1, a_2, a_3, \ldots a_n$ will give n particular values of v_x, a_1^x, a_2^x, $a_3^x \ldots a_n^x$ which satisfy equation (1); therefore its complete integral is

$$v_x = c_1 a_1^x + c_2 a_2^x + \ldots + c_n a_n^x;$$

and the complete integral of the proposed equation is

$$u_x = c_1 a_1^x + c_2 a_2^x + \ldots + c_n a_n^x + \frac{q}{f(1)}.$$

This shews that we may treat the proposed equation as if it had no term q independent of u_x, provided we divide that term by $f(1)$ the sum of the coefficients, and add the quotient to the value of u_x obtained on the supposition that $q = 0$.

If we replace c_1 by $c_0 - q \div f(1)$, and a_1 by $1 + h$ where h is small, so that very nearly $a_1^x = 1 + hx$, and $f(1) = -hf'(1)$ since $1 + h$ is a root of $f(a) = 0$, then reducing and making $h = 0$, we get the form which the solution takes when $f(1) = 0$, viz.

$$u_x = c_0 + \frac{qx}{f'(1)} + \&c.$$

78. If the auxiliary equation have equal roots, the above ceases to be the form of the complete solution; because, in that case, it does not involve the due number of arbitrary constants; and it must be modified as follows. Suppose two roots a_1, a_2, to be very nearly equal to one another, so that $a_2 = a_1 + h$, h being a very small known quantity; then

$$c_1 a_1^x + c_2 a_2^x = (c_1 + c_2)\, a_1^x + c_2 \left\{x a_1^{x-1} h + \frac{x(x-1)}{1\,.\,2}\, a_1^{x-2} h^2 + \&c.\right\}$$

$$= C_1 a_1^x + C_2 \left\{x a_1^x + \frac{x(x-1)}{1\,.\,2}\, a_1^{x-1} h + \&c.\right\},$$

replacing the constants $c_1 + c_2$ by C_1, and $\dfrac{hc_2}{a_1}$ by C_2;

now this equation continues true however small h be taken, and therefore when $h = 0$; in which case the second member becomes $(C_1 + C_2 x)\, a_1^x$;

$$\therefore\ u_x = (C_1 + C_2 x)\, a_1^x + c_3 a_3^x + \&c.$$

Similarly, if the auxiliary equation have r roots equal to a_1, the complete solution will be

$$u_x = (c_0 + c_1 x + c_2 x^2 + \ldots + c_{r-1} x^{r-1})\, a_1{}^x + c_{r+1} a^x{}_{r+1} + \&c\,;$$

of the correctness of which, we may be assured by the following reverse process. Assume $u_x = a^x w_x$, then

$$u_{x+n} = a^{x+n} \,.\, w_{x+n} = a^x \,.\, a^n\, (1 + \Delta)^n w_x = a^x\, (a + a\Delta)^n w_x,$$

and the first side of the proposed equation becomes, when divided by a^x,

$$= \{(a + a\Delta)^n + p_1\, (a + a\Delta)^{n-1} + \&c. + p_n\}\, w_x$$

$$= f(a + a\Delta)\, w_x = \{f(a) + f'(a) \,.\, a\Delta + f''(a)\, \frac{a^2\Delta^2}{1\,.\,2} + \&c.\}\, w_x$$

$$= f(a)\, w_x + \frac{a}{1} \,.\, f'(a) \,.\, \Delta w_x + \frac{a^2}{1\,.\,2} f''(a) \,.\, \Delta^2 w_x + \&c. + a^n \Delta^n w_x.$$

Now suppose $a = a_1$, and $f(a) = 0$ to have r roots equal to a_1; this makes the terms as far as $f^{(r-1)}(a)$ vanish; and if

$$w_x = c_0 + c_1 x + \ldots + c_{r-1} x^{r-1}, \text{ then } \Delta^r w_x = 0,\ \Delta^{r+1} w_x = 0, \&c.,$$

and all the remaining terms vanish; and consequently the equation is satisfied by

$$u_x = (c_0 + c_1 x + \ldots + c_{r-1} x^{r-1})\, a_1{}^x.$$

Hence we see that every root a_1 that occurs r times in the auxiliary equation, gives rise, in the complete integral, to a term of the form $f_{r-1}(x) a_1{}^x$, where $f_{r-1}(x)$ denotes a rational integral function of x of the $(r-1)^{\text{th}}$ degree involving r arbitrary constants. The root a_1 may be either real or imaginary.

79. Also if the auxiliary equation have a pair of imaginary roots,

$$m \pm n\sqrt{-1} = \rho\, (\cos\theta \pm \sqrt{-1}\sin\theta),$$

$$\text{putting} \quad \rho = \sqrt{m^2 + n^2}, \quad \tan\theta = \frac{n}{m},$$

the corresponding terms in the valne of u_x will be

$$C\rho^x\, (\cos\theta + \sqrt{-1}\sin\theta)^x + C'\rho^x\, (\cos\theta - \sqrt{-1}\sin\theta)^x$$
$$= \rho^x\, (c_1 \cos x\theta + c_2 \sin x\theta),$$

changing the arbitrary constants.

And if the same pair of imaginary roots occur r times in the auxiliary equation, the corresponding terms in the value of u_x will be

$$(a_0 + a_1 x + \dots + a_{r-1} x^{r-1}) \rho^x (\cos\theta + \sqrt{-1}\sin\theta)^x$$
$$+ (b_0 + b_1 x + \dots + b_{r-1} x^{r-1}) \rho^x (\cos\theta - \sqrt{-1}\sin\theta)^x;$$

or changing the arbitrary constants,

$$(c_0 + c_1 x + \dots + c_{r-1} x^{r-1}) \rho^x \cos x\theta + (c'_0 + c'_1 x + \dots + c'_{r-1} x^{r-1}) \rho^x \sin x\theta.$$

Ex. 1. $u_{x+4} - 2u_{x+3} - 13u_{x+2} + 14u_{x+1} + 24u_x = 0.$

The auxiliary equation is $(a+1)(a-2)(a+3)(a-4) = 0$;

$$\therefore\ u_x = c_1(-1)^x + c_2 2^x + c_3(-3)^x + c_4 4^x.$$

Ex. 2. $u_{x+3} - 5u_{x+2} + 8u_{x+1} - 4u_x = 0.$

The auxiliary equation is $(a-1)(a-2)^2 = 0$;

$$\therefore\ u_x = c_0 + (c + c'x)\, 2^x;$$

and if the second member be q,

$$u_x = c_0 + qx + (c + c'x)\, 2^x.$$

Ex. 3. $u_{x+3} - u_x = q.$

The auxiliary equation is $a^3 - 1 = 0$, whose roots are 1, and $\cos\frac{2\pi}{3} \pm \sqrt{-1}\sin\frac{2\pi}{3}$;

$$\therefore\ u_x = c_0 + c\cos\frac{2\pi x}{3} + c'\sin\frac{2\pi x}{3} + \tfrac{1}{3}qx.$$

80.* The Differential Calculus being a particular case of that of Finite Differences, a strict analogy exists between the methods and results in the two Subjects, as the reader cannot fail to have observed; indeed whenever in the latter a result is obtained with an indeterminate increment for the principal variable, it is possible to pass to the corresponding result in the former, by a method similar to that pursued in Art. 25; and which we shall further illustrate by the following instance.

In the equation $u_{x+2h} - 2mu_{x+h} + (m^2 + n^2)\, u_x = 0$,

putting $u_x = a^x$, we find $a = (m \pm n\sqrt{-1})^{\frac{1}{h}}$;

$$\therefore\ u_x = C_1 (m + n\sqrt{-1})^{\frac{x}{h}} + C_2 (m - n\sqrt{-1})^{\frac{x}{h}}$$

$$= (m^2 + n^2)^{\frac{x}{2h}} \left\{c_1 \cos\left(\frac{x}{h} \tan^{-1}\frac{n}{m}\right) + c_2 \sin\left(\frac{x}{h}\tan^{-1}\frac{n}{m}\right)\right\}.$$

Now the proposed equation is

$$\Delta^2 u_x - 2(m-1)\,\Delta u_x + \{(m-1)^2 + n^2\}\, u_x = 0;$$

or, if we replace the known quantities $m - 1$ and n by other known quantities $m_1 h$ and $n_1 h$, the equation and its solution take the forms

$$\frac{\Delta^2 u_x}{h^2} - 2m_1 \frac{\Delta u_x}{h} + (m_1^2 + n_1^2)\, u_x = 0,$$

$$u_x = \{(1 + m_1 h)^2 + n_1^2 h^2\}^{\frac{x}{2h}}$$

$$\times \left\{c_1 \cos\left(\frac{x}{h}\tan^{-1}\frac{n_1 h}{1 + m_1 h}\right) + c_2 \sin\left(\frac{x}{h}\tan^{-1}\frac{n_1 h}{1 + m_1 h}\right)\right\}.$$

Now take the limits of these expressions when $h = 0$, and we find the well-known results, since $(1 + 2m_1 h)^{\frac{1}{2h}}$ becomes e^{m_1},

$$\frac{d^2 u_x}{dx^2} - 2m_1 \frac{du_x}{dx} + (m_1^2 + n_1^2)\, u_x = 0,$$

$$u_x = e^{m_1 x}(c_1 \cos n_1 x + c_2 \sin n_1 x).$$

80. Before proceeding to the general case of linear equations, we shall consider the case of a linear equation with all its coefficients constant, but having a function of x, X, for the term independent of u_x.

This equation can be shortly treated by the method of separation of symbols. For if we substitute $D^n u_x$ for u_{x+n}, it becomes

$$(D^n + p_1 D^{n-1} + p_2 D^{n-2} + \ldots + p_n)\, u_x = X;$$

or, if $a_1, a_2, \ldots a_n$ be the roots of the auxiliary equation,

$$(D-a_1)(D-a_2)\dots(D-a_n)\,u_x = X;$$

$$\therefore\; u_x = \frac{1}{(D-a_1)(D-a_2)\dots(D-a_n)}\,X\dots\dots\dots\dots\,(1).$$

Now resolve this function of D into partial fractions (see Integral Calculus, Art. 34), so that the values of A_1, A_2, &c., are all known in terms of a_1, a_2, &c. respectively; then

$$u_x = \left(\frac{A_1}{D-a_1} + \frac{A_2}{D-a_2} + \dots + \frac{A_n}{D-a_n}\right) X.$$

But we have seen Art. 58, that $(D-a)^{-1}X = a^{x-1}\Sigma\,(Xa^{-x}) + Ca^{x-1}$.

$$\therefore\; u_x = A_1 a_1^{x-1}\Sigma\,(Xa_1^{-x}) + A_2 a_2^{x-1}\Sigma\,(Xa_2^{-x}) + \dots + A_n a_n^{x-1}\Sigma\,(Xa_n^{-x})$$
$$+ c_1 a_1^{x} + c_2 a_2^{x} + \dots + c_n a_n^{x};$$

an arbitrary constant being introduced by each of the integrations, and the product of two or more constants being replaced by a single constant. The part of the above expression involving the arbitrary constants is called the complementary function; and it is the value of u_x which satisfies the proposed equation when $X = 0$.

81. If the auxiliary equation have r roots equal to a_1, then from (1)

$$u_x = \frac{1}{(D-a_1)^r (D-a_{r+1})\dots(D-a_n)}\,X;$$

and this function of D may be resolved into partial fractions, (see Integral Calculus, Art. 36) so that

$$u_x = \left\{\frac{A_1}{D-a_1} + \frac{A_2}{(D-a_1)^2} + \dots + \frac{A_r}{(D-a_1)^r}\right.$$
$$\left. + \frac{A_{r+1}}{D-a_{r+1}} + \dots + \frac{A_n}{D-a_n}\right\} X.$$

But $(D-a)^{-n} X = a^{x-n}\Sigma^n\,(a^{-x}X) + a^{x-n}\,(C_0 + C_1 x + \dots + C_{n-1}x^{n-1})$,

C_0, C_1, &c. being the constants introduced after each integration;

$$\therefore\; u_x = A_1 a_1^{x-1}\Sigma\,(a_1^{-x}X) + A_2 a_1^{x-2}\Sigma^2\,(a_1^{-x}X) + \dots + A_r a_1^{x-r}\Sigma^r\,(a_1^{-x}X$$
$$+ A_{r+1} a_{r+1}^{x-1}\Sigma\,(a_{r+1}^{-x}X) + \dots + A_n a_n^{x-1}\Sigma\,(a_n^{-x}X)$$
$$+ (c_0 + c_1 x + \dots + c_{r-1}x^{r-1})\,a_1^{x} + c_{r+1}a_{r+1}^{x} + \dots + c_n a_n^{x},$$

a single constant, as before, being substituted for the sum or product of several constants in forming the complementary function.

82. Again, suppose $a_1 = \rho(\cos\theta + \sqrt{-1}\sin\theta)$ to be an imaginary root of the auxiliary equation; then since $\frac{A_1}{a_1}$ is a function of a_1, it will be of the form $R(\cos\alpha + \sqrt{-1}\sin\alpha)$; and the term involving a_1 in the value of u_x will consequently be

$$R\rho^x\{\cos(x\theta+\alpha)+\sqrt{-1}\sin(x\theta+\alpha)\}\Sigma\rho^{-x}X(\cos x\theta-\sqrt{-1}\sin x\theta)\ldots(1).$$

Now the term in u_x involving the conjugate root to a_1, will result from this, by changing the sign of $\sqrt{-1}$; and therefore the sum of the two terms introduced into the value of u_x by the pair of imaginary roots $\rho(\cos\theta \pm \sqrt{-1}\sin\theta)$ will equal twice the real part of expression (1); that is,

$$2R\rho^x\cos(x\theta+\alpha)(\Sigma\rho^{-x}X\cos x\theta + C)$$
$$+ 2R\rho^x\sin(x\theta+\alpha)(\Sigma\rho^{-x}X\sin x\theta + C');$$

where

$$2R\rho^x C\cos(x\theta+\alpha),\ 2R\rho^x C'\sin(x\theta+\alpha)$$

are the terms introduced into the complementary function; and, if we alter the constants, may be replaced by

$$\rho^x(c\cos x\theta + c'\sin x\theta).$$

83. Exactly in the same way, if the imaginary root

$$a_1 = \rho(\cos\theta + \sqrt{-1}\sin\theta),$$

occur r times in the auxiliary equation it will produce in the value of u_x, r terms of the form $A_m a_1^{x-m}\Sigma^m(a_1^{-x}X)$; or, since $\frac{A^m}{a_1^m}$ is a function of a_1 and may be assumed

$$= R_m(\cos\alpha_m + \sqrt{-1}\sin\alpha_m),$$

of the form

$$R_m\rho^x\{\cos(x\theta+\alpha_m)$$
$$+\sqrt{-1}\sin(x\theta+\alpha_m)\}\Sigma^m\rho^{-x}(\cos x\theta - \sqrt{-1}\sin x\theta)X\ldots(2).$$

But the root conjugate to a_1 will produce a term exactly the

same as this, except with $-\sqrt{-1}$ instead of $+\sqrt{-1}$; consequently the sum of these terms will equal twice the real part of (2); that is,

$$2R_m\rho^x\{\cos(x\theta+\alpha_m)\,\Sigma^m(\rho^{-x}\cos x\theta X)+\sin(x\theta+\alpha_m)\,\Sigma^m(\rho^{-x}\sin x\theta X)\},$$

and to get all the terms introduced by the pair of imaginary roots $\rho\,(\cos\theta\pm\sqrt{-1}\sin\theta)$ that occur r times in the auxiliary equation, m in the above formula must receive all values from 1 to r. Also we see that the part of the complementary function introduced by these roots, by substituting a single constant for the sum or product of other constants, will take the form

$$(c_0+c_1x+\ldots+c_{r-1}x^{r-1})\,\rho^x\cos x\theta+(c'_0+c'_1x+\ldots+c'_{r-1}x^{r-1})\,\rho^x\sin x\theta.$$

We shall now give an instance of each of the cases that have been examined. It may be observed, that for equations capable of being reduced to either of the forms

$$f(D)\,u_x=a^x,\quad\text{or } f(D)\,u_x=p_0x^m+p_1x^{m-1}+\&\text{c.}+p_m,$$

the process may be greatly simplified. For in the former case, since $Da^x=a\,.\,a^x$, the symbol D is equivalent to the factor a, and may have a substituted for it in the formula $u_x=\dfrac{1}{f(D)}\,a^x$; so that the solution of the equation $f(D)\,u_x=a^x$ is $u_x=\dfrac{a^x}{f(a)}+$ complementary function. And in the latter case, if $\dfrac{1}{f(D)}=\dfrac{1}{f(1+\Delta)}$ can be expanded in the form $A_0+A_1\Delta+A_2\Delta^2+\&\text{c.}$, then

$$u_x=(A_0+A_1\Delta+A_2\Delta^2+\ldots+A_m\Delta^m)\,(p_0x^m+p_1x^{m-1}+\ldots+p_m)$$

+ complementary function.

Ex. 1. $u_{x+3}-9u_{x+2}+26u_{x+1}-24u_x=5^x$,

$$\text{or }(D^3-9D^2+26D-24)\,u_x=f(D)\,u_x=5^x;$$

$$\therefore\ u_x=\frac{5^x}{(D-2)(D-3)(D-4)}=\left(\frac{1}{2}\frac{1}{D-2}-\frac{1}{D-3}+\frac{1}{2}\frac{1}{D-4}\right)5^x$$

$$=\frac{1}{2}\,.\,2^{x-1}\Sigma\left(\frac{5}{2}\right)^x-3^{x-1}\Sigma\left(\frac{5}{3}\right)^x+\frac{1}{2}\,.\,4^{x-1}\Sigma\left(\frac{5}{4}\right)^x$$

$$= 2^{x-2}\frac{2}{3}\left(\frac{5}{2}\right)^x - 3^{x-1}\frac{3}{2}\left(\frac{5}{3}\right)^x + \frac{1}{2}\,.\,4^{x-1}\,4\left(\frac{5}{4}\right)^x + \&c.$$

$$= \frac{5^x}{6} + c_0 2^x + c_1 3^x + c_2 4^x,$$

a result we could have foreseen because $f(5) = 3\,.\,2\,.\,1 = 6$.

Ex. 2. $a^2 u_{x+2} - 2au_{x+1} + u_x = a^2 x$,

$$\text{or } (aD-1)^2 u_x = a^2 x;$$

$$\therefore\ u_x = \left(D - \frac{1}{a}\right)^{-2} x = \frac{1}{a^{x-2}}\Sigma^2 (xa^x) = \frac{1}{a^{x-2}}(x\Sigma^2 a^x - 2\Sigma^3 a^{x+1})$$

$$= \frac{1}{a^{x-2}}\left\{\frac{xa^x}{(a-1)^2} - \frac{2a^{x+1}}{(a-1)^3} + Cx + C'\right\}$$

$$= \frac{a^2 x}{(a-1)^2} - \frac{2a^3}{(a-1)^3} + (c + c'x)\frac{1}{a^x}.$$

Otherwise,

$$u_x = \left(1 - \frac{1}{a} + \Delta\right)^{-2} x = \left\{\left(1 - \frac{1}{a}\right)^{-2} - 2\left(1 - \frac{1}{a}\right)^{-3}\Delta + \&c.\right\} x$$

$$= \frac{a^2 x}{(a-1)^2} - \frac{2a^3}{(a-1)^3} + (c + c'x)\,a^{-x}.$$

Ex. 3. $u_{x+2} - 2a\cos\theta u_{x+1} + a^2 u_x = X$,

$$\text{or } (D^2 - 2a\cos\theta D + a^2)\,u_x = X;$$

$$\therefore\ u_x = \frac{X}{(D-az)(D-az^{-1})} = \frac{1}{a(z-z^{-1})}\left(\frac{X}{D-az} - \frac{X}{D-az^{-1}}\right),$$

putting $2\cos\theta = z + z^{-1}$, and consequently $2\sqrt{-1}\sin\theta = z - z^{-1}$;

$$\therefore\ u_x = \frac{1}{a(z-z^{-1})}\{(az)^{x-1}\Sigma X(az)^{-x} - (az^{-1})^{x-1}\Sigma X(a^{-1}z)^x\}.$$

Now the value of the former of these integrals is

$$\frac{a^{x-2}}{2\sqrt{-1}\sin\theta}\{\cos(x-1)\,\theta$$

$$+ \sqrt{-1}\sin(x-1)\,\theta\}\,\Sigma X a^{-x}(\cos x\theta - \sqrt{-1}\sin x\theta);$$

hence, taking twice the possible part of this expression, we have

$$u_x = \frac{a^{x-2}}{\sin\theta}\{\sin(x-1)\theta\Sigma(Xa^{-x}\cos x\theta) - \cos(x-1)\theta\Sigma(Xa^{-x}\sin x\theta)\},$$

the complementary function being

$$a^x(c\cos x\theta + c'\sin x\theta).$$

If we suppose $X = a^x$, then as we could have foreseen

$$u_x = \frac{a^{x-2}}{2(1-\cos\theta)} + a^x(c\cos x\theta + c'\sin x\theta).$$

The case of $\theta = 0$ will be presently noticed.

Ex. 4. $u_{x+4} + p_1 u_{x+3} + p_2 u_{x+2} + p_3 u_{x+1} + p_4 u_x = X,$

where the auxiliary equation is supposed to have two pairs of imaginary roots of the form $a(\cos\theta \pm \sqrt{-1}\sin\theta)$, and consequently the proposed equation is of the form

$$(D^2 - 2a\cos\theta D + a^2)^2 u_x = (D - az)^2 (D - az^{-1})^2 u_x = X.$$

Now by the preceding example we have

$$\frac{2a\sqrt{-1}\sin\theta}{(D-az)(D-az^{-1})} = \frac{1}{D-az} - \frac{1}{D-az^{-1}};$$

$$\therefore \frac{-4a^2\sin^2\theta}{(D-az)^2(D-az^{-1})^2} = \frac{1}{(D-az)^2}$$

$$+\frac{\sqrt{-1}}{a\sin\theta}\left(\frac{1}{D-az} - \frac{1}{D-az^{-1}}\right) + \frac{1}{(D-az^{-1})^2};$$

$$\therefore -4a^2\sin^2\theta u_x = (az)^{x-2}\Sigma^2(az)^{-x}X$$

$$+\frac{\sqrt{-1}}{a\sin\theta}(az)^{x-1}\Sigma(az)^{-x}X + \text{terms in } az^{-1}$$

$$= a^{x-2}\{\cos(x-2)\theta + \sqrt{-1}\sin(x-2)\theta\}$$

$$\times \Sigma^2 a^{-x} X(\cos x\theta - \sqrt{-1}\sin x\theta)$$

$$+\frac{a^{x-2}\sqrt{-1}}{\sin\theta}\{\cos(x-1)\theta + \sqrt{-1}\sin(x-1)\theta\}$$

$$\times \Sigma a^{-x} X(\cos x\theta - \sqrt{-1}\sin x\theta) + \&c.,$$

and taking in these terms only the possible part, and doubling it, we have $-2\sin^3\theta a^{-x+4} u_x$

$$= \sin\theta\cos(x-2)\,\theta\Sigma^2(a^{-x}X\cos x\theta)$$

$$+\sin\theta\sin(x-2)\,\theta\Sigma^2(a^{-x}X\sin x\theta)$$

$$+\cos(x-1)\,\theta\Sigma(a^{-x}X\sin x\theta)-\sin(x-1)\,\theta\Sigma(a^{-x}X\cos x\theta),$$

the complementary function being

$$(b+b'x)\,a^x\sin x\theta+(c+c'x)\,a^x\cos x\theta.$$

If $X=a^x$, the result will be, as we are aware,

$$u_x=\frac{a^{x-4}}{4(1-\cos\theta)^2}+\text{complementary function.}$$

84. To integrate the linear equation of differences of the n^{th} order, the coefficients being functions of x,

$$u_{x+n}+p_1u_{x+n-1}+p_2u_{x+n-2}+\ldots+p_nu_x=X,$$

on the supposition that it can be solved when $X=0$.

If z_x, ${}^1z_x$, ... ${}^{n-1}z_x$ be n particular values of v_x in the equation

$$v_{x+n}+p_1v_{x+n-1}+p_2v_{x-n-2}+\ldots+p_nv_x=0\ldots\ldots\ldots\ldots\ldots(1),$$

with which the proposed coincides when its second member is zero, we have (Art. 77)

$$v_x=c_0z_x+c_1{}^1z_x+c_2{}^2z_x+\ldots+c_{n-1}{}^{n-1}z_x.$$

If we now divide both sides by z_x and take the difference, we shall eliminate c_0; next dividing both sides by the coefficient of c_1 in the new result, which suppose y_x, and taking the difference, we shall eliminate c_1; again dividing by w_x the coefficient of c_2 and taking the difference, we shall eliminate c_2; and proceeding in this manner till all the constants are eliminated, our final result will be of the form (each Δ affecting the whole of the expression that follows it)

$$\Delta\frac{1}{t_x}\Delta\frac{1}{w_x}\ldots\Delta\frac{1}{y_x}\Delta\left(\frac{v_x}{z_x}\right),$$

in which expression the coefficient of v_{x+n} is evidently

$$\frac{1}{t_{x+1}\, w_{x+2} \cdots y_{x+n-1}\, z_{x+n}};$$

therefore, dividing by this coefficient, we get the expression

$$t_{x+1}\, w_{x+2} \cdots y_{x+n-1}\, z_{x+n}\, \Delta \frac{1}{t_x} \Delta \frac{1}{w_x} \cdots \Delta \frac{1}{y_x} \Delta \left(\frac{v_x}{z_x}\right),$$

which must be equivalent to the first member of equation (1). Therefore the same expression, only with u_x instead of v_x, must be equivalent to the first member of the proposed equation, and consequently equal to X; hence, equating these equals, and integrating, we get

$$u_x = z_x \Sigma y_x \Sigma w_x \ldots \Sigma t_x \Sigma \frac{X}{t_{x+1}\, w_{x+2} \cdots y_{x+n-1}\, z_{x+n}},$$

(each Σ affecting the whole of the expression which follows it) which is a general formula for the integration of any linear equation of differences whose solution can be effected when $X=0$.

In the case of constant coefficients if the auxiliary equation contain equal or imaginary roots, this method is still applicable; it is only necessary to assume for v_x the value belonging to the case of equal or imaginary roots, as will be seen in some of the following instances.

Ex. 1. Suppose the coefficients of the equation to be constant, and $a_1, a_2, \ldots a_n$ to be the n roots of its auxiliary equation.

Then $v_x = c_1 a^x{}_1 + c_2 a^x{}_2 + c_3 a^x{}_3 + \ldots + c_n a^x{}_n$,

$$\therefore\ \Delta \frac{v_x}{a^x{}_1} = c_2 \left(\frac{a_2}{a_1}\right)^x + c_3 \left(\frac{a_3}{a_1}\right)^x + \ldots + c_n \left(\frac{a_n}{a_1}\right)^x,$$

changing, both in this, and in the similar succeeding steps, the arbitrary constants;

$$\Delta \left(\frac{a_1}{a_2}\right)^x \Delta \frac{v_x}{a^x{}_1} = c_3 \left(\frac{a_3}{a_2}\right)^x + \ldots + c_n \left(\frac{a_n}{a_2}\right)^x$$

$$\Delta\left(\frac{a_2}{a_3}\right)^x \Delta\left(\frac{a_1}{a_2}\right)^x \Delta\frac{v_x}{a^x_{\ 1}} = c_4\left(\frac{a_4}{a_3}\right)^x + \ldots + c_n\left(\frac{a_n}{a_3}\right)^x$$

....

$$\Delta\left(\frac{a_{n-1}}{a_n}\right)^x \Delta\left(\frac{a_{n-2}}{a_{n-1}}\right)^x \ldots \Delta\left(\frac{a_1}{a_2}\right)^x \Delta\frac{v_x}{a^x_{\ 1}} = 0,$$

in which expression the coefficient of v_{x+n} is evidently

$$\frac{1}{a^x_{\ n}}\,\frac{1}{a_n a_{n-1} \ldots a_2 a_1} = \frac{1}{(-1)^n p_n a^x_{\ n}},$$

therefore, dividing by this coefficient, and replacing v_x by u_x, we get

$$(-1)^n p_n a^x_{\ n} \Delta\left(\frac{a_{n-1}}{a_n}\right)^x \Delta\left(\frac{a_{n-2}}{a_{n-1}}\right)^x \ldots \Delta\left(\frac{a_1}{a_2}\right)^x \Delta\frac{u_x}{a^x_{\ 1}} = X,$$

$$\therefore\ u_x = \frac{(-1)^n}{p_n}\,a^x_{\ 1}\Sigma\left(\frac{a_2}{a_1}\right)^x \ldots \Sigma\left(\frac{a_n}{a_{n-1}}\right)^x \Sigma\left(\frac{X}{a^x_n}\right);$$

or, if we choose to introduce the arbitrary constant after each integration,

$$u_x = c_1 a^x_{\ 1} + c_2 a^x_{\ 2} + \ldots c_n a^x_{\ n},$$

$$+ (-1)^n \frac{1}{p_n} a^x_{\ 1}\Sigma\left(\frac{a_2}{a_1}\right)^x \Sigma\left(\frac{a_3}{a_2}\right)^x \ldots \Sigma\left(\frac{a_n}{a_{n-1}}\right)^x \Sigma\left(\frac{X}{a^x_{\ n}}\right).$$

This result may be readily obtained by separation of symbols, and so shewn to agree with that obtained in Art. 80. For the proposed equation is

$$D^n u_x + p_1 D^{n-1} u_x + \ldots + p_{n-1} D u_x + p_n u_x = X,$$

$$\text{or}\ \ (D - a_n)(D - a_{n-1}) \ldots (D - a_1)\, u_x = X;$$

$$\therefore\ (D - a_{n-1})(D - a_{n-2}) \ldots (D - a_1)\, u_x = (D - a_n)^{-1} X = a_n^{\,x-1}\Sigma\,(a_n^{-x} X)$$

$$= a^x_{\ n} X_n \ \text{ suppose, where } X_n = \frac{1}{a_n}\Sigma\,(a_n^{\ -x} X);$$

$$\therefore\ (D - a_{n-2})(D - a_{n-3}) \ldots (D - a_1)\, u_x = (D - a_{n-1})^{-1} a^x_{\ n} X_n$$

$$= a_{n-1}^{\ \ x-1}\Sigma\left(\frac{a_n}{a_{n-1}}\right)^x X_n = a^x_{\ n-1} X_{n-1},\ \text{ where } X_{n-1} = \frac{1}{a_{n-1}}\Sigma\left(\frac{a_n}{a_{n-1}}\right)^x X_n,$$

and so on till we arrive at

$$u_x = (D - a_1)^{-1} a^x{}_2 X_2 = a_1^{x-1} \Sigma \left(\frac{a_2}{a_1}\right)^x X_2, \text{ when } X_2 = \frac{1}{a_2} \Sigma \left(\frac{a_3}{a_2}\right)^x X_3.$$

Hence restoring the values of $X_2, X_3 \dots X_n$, we get

since $a_1 a_2 \dots a_n = (-1)^n p_n$,

$$u_x = (-1)^n \frac{1}{p_n} a^x{}_1 \Sigma \left(\frac{a_2}{a_1}\right)^x \Sigma \left(\frac{a_3}{a_2}\right)^x \dots \Sigma \left(\frac{a_n}{a_{n-1}}\right)^x \Sigma \left(\frac{X}{a^x{}_n}\right),$$

where each Σ affects the whole of the expression that follows it.

Ex. 2. $$a^2 u_{x+2} - 2a u_{x+1} + u_x = a^2 X.$$

$$\text{Here } v_x = (c + c^1 x) \frac{1}{a_x}; \quad \therefore \Delta^2 (a^x v_x) = 0,$$

in which expression, the coefficient of v_{x+2} is a^{x+2}; therefore, dividing by this quantity, and replacing v_x by u_x, we get

$$\frac{1}{a^{x+2}} \Delta^2 (a^x u_x) = X; \quad \therefore u_x = \frac{1}{a^x} \Sigma^2 (X a^{x+2}).$$

Ex. 3. $u_{x+2} - 2m u_{x+1} + (m^2 + n^2) u_x = X.$

Here $v_x = c_1 \rho^x \sin x\theta + c_2 \rho^x \cos x\theta$,

$$\text{where } \rho^2 = m^2 + n^2, \quad \tan\theta = \frac{n}{m};$$

$$\therefore \frac{v_x}{\rho^x \cos x\theta} = c_1 \tan x\theta + c_2;$$

$$\therefore \Delta \left(\frac{v_x}{\rho^x \cos x\theta}\right) = \frac{c_1 \sin\theta}{\cos x\theta \cos (x+1)\theta};$$

$$\therefore \Delta \cos x\theta \cos (x+1)\theta \Delta \left(\frac{v_x}{\rho^x \cos x\theta}\right) = 0,$$

in which expression, the coefficient of v_{x+2} is $\dfrac{\cos (x+1)\theta}{\rho^{x+2}}$; therefore, dividing by this coefficient, and replacing v_x by u_x, we get

$$\frac{\rho^{x+2}}{\cos (x+1)\theta} \Delta \cos x\theta \cos (x+1)\theta \Delta \left(\frac{u_x}{\rho^x \cos x\theta}\right) = X;$$

$$\therefore \; u_x = \rho^x \cos x\theta \Sigma \sec x\theta \sec (x+1)\,\theta \Sigma \left\{\frac{X \cos (x+1)\,\theta}{\rho^{x+2}}\right\};$$

to which we may add the terms $c_1\rho^x \sin x\theta + c_2\rho^x \cos x\theta$, if we suppose a constant to be added after each integration.

84*. Having given a particular integral of the equation

$$u_{x+2} + A_x u_{x+1} + B_x u_x = 0,$$

to find its complete solution.

Let v_x be a particular integral, then

$$v_{x+2} + A_x v_{x+1} + B_x v_x = 0.$$

Hence, eliminating A_x, we find, putting $\dfrac{u_x}{v_x} = z_x$,

$$\frac{\Delta z_{x+1}}{\Delta z_x} = \frac{B_x v_x}{v_{x+2}} = w_x, \text{ suppose};$$

$$\text{or } \; \Delta \log (\Delta z_x) = \log w_x,$$

$$\therefore \; \Delta z_x = CPw_{x-1}, \quad \text{(Art. 50)}$$

$$\therefore \; u_x = Cv_x \Sigma (Pw_{x-1}).$$

85. If we know a particular integral of a linear equation of any order that has no term independent of u_x, we may reduce it to another equation of the same kind of the order immediately inferior.

Let $u_x = v_x$ be a particular integral of a linear equation of differences of the n^{th} order reduced to the form

$$\Delta^n u_x + q_1 \Delta^{n-1} u_x + q_2 \Delta^{n-2} u_x + \ldots + q_n u_x = f(\Delta)\, u_x = 0 \ldots\ldots (1).$$

Assume $u_x = v_x \Sigma w_x$; then

$$\Delta^n u_x = (\Delta + D\Delta')^n v_x \Sigma w_x,$$

where Δ and D affect v_x only, and Δ' affects Σw_x only (Art. 34), and the proposed equation becomes

$$f(\Delta + D\Delta')\, v_x \Sigma w_x = 0.$$

Now $f(\Delta + D\Delta')$ is a rational integral function of Δ and Δ', which we wish to arrange according to powers of Δ', and this may be done at once by Taylor's theorem, which gives

$$f(\Delta) + f_1(\Delta)\, D\Delta' + \frac{1}{1\,.\,2} f_2(\Delta)\, D^2\Delta'^2 + \dots + D^n\Delta'^n.$$

Hence, observing that $f(\Delta)\, v_x = 0$, since v_x substituted for u_x satisfies the proposed equation, we get the depressed equation

$$w_x f_1(\Delta)\, v_{x+1} + \frac{1}{1\,.\,2}\,\Delta w_x f_2(\Delta)\, v_{x+2} + \dots + \Delta^{n-1} w_x \,.\, v_{x+n} = 0\,;$$

or, reversing the order of the terms, {since $f_r(\Delta)$ means the same function of Δ, that $\dfrac{d^r f(x)}{dx^r}$ does of x},

$$v_{x+n}\,\Delta^{n-1} w_x + (n\Delta v_{x+n-1} + q_1 v_{x+n-1})\,\Delta^{n-2} w_x$$

$$+ \left\{\frac{n(n-1)}{1\,.\,2}\,\Delta^2 v_{x+n-2} + (n-1)\, q_1 \Delta v_{x+n-2} + q_2 v_{x+n-2}\right\} \Delta^{n-3} w_x + \dots$$

$$+ \{n\Delta^{n-1} v_{x+1} + (n-1)\, q_1 \Delta^{n-2} v_{x+1} + \dots + q_{n-1} v_{x+1}\}\, w_x = 0 \dots\dots (2),$$

a linear equation of the $(n-1)^{\text{th}}$ order, of the same form as the original one.

Similarly, if we know another particular value of u_x, z_x, then $\Delta\left(\dfrac{z_x}{v_x}\right)$ will be a value of w_x in equation (2), which may be depressed to another of the same form of the $(n-2)^{\text{th}}$ order; and if we know r particular solutions of equation (1), we may in this way depress it to an equation of the same form of the $(n-r)^{\text{th}}$ order. As a linear equation of differences of the first order and degree can always be solved, it appears that to obtain the complete integral of a linear equation of the n^{th} order, we must know $n-1$ particular solutions.

Equations of Differences which admit of being reduced to Linear Equations. Simultaneous Equations.

86. Besides linear equations of differences, and such equations as can be reduced to that form, very little is known of equations of differences of the second and higher orders and

degrees. Where such equations can be solved, it is generally by means of particular substitutions adapted to the form of the proposed equation; or by transformations of a more intricate character. The following are instances of equations which admit of being reduced to linear equations with constant coefficients, by particular substitutions.

1. $$u_{x+n} + p_1 v_{x+n} u_{x+n-1} + p_2 v_{x+n} v_{x+n-1} u_{x+n-2} + \dots + p_n v_{x+n} v_{x+n-1} \dots v_{x+1} u_x = 0;$$

where p_1, p_2, &c. are constants, and v_x is any function of x. Assume $u_x = w_x P v_x$, then the equation becomes divisible by $P v_{x+n}$, and is reduced to the linear equation with constant coefficients,

$$w_{x+n} + p_1 w_{x+n-1} + p_2 w_{x+n-2} + \dots + p_n w_x = 0.$$

Ex. $u_{x+2} - 2(x+2)\, u_{x+1} - 3(x+2)(x+1)\, u_x = 0.$

$$u_x = \{a 3^x + b(-1)^x\} \times 1 \,.\, 2 \,.\, 3 \dots x.$$

2. $$u_{x+n} + p_1 a^x u_{x+n-1} + p_2 a^{2x} u_{x+n-2} + \dots + p_n a^{nx} u_x = 0.$$

Assume $u_x = v_x a^{(1+2+3\dots+x)}$, then any term $p_r a^{rx} u_{x+n-r}$ becomes

$$= p_r v_{x+n-r} a^{\frac{1}{2}x(x+2n+1)} \,.\, a^{\frac{1}{2}(n-r)(n-r+1)};$$

therefore the equation becomes divisible by $a^{\frac{1}{2}x(x+2n+1)}$, and is reduced to the linear equation with constant coefficients,

$$a^{\frac{1}{2}(n+1)n} v_{x+n} + p_1 a^{\frac{1}{2}n(n-1)} v_{x+n-1} + p_2 a^{\frac{1}{2}(n-1)(n-2)} v_{x+n-2} + \dots + p_n v_x = 0.$$

3. $$u_{x+2} + \{a + b(-1)^x\}\, u_{x+1} + c u_x = 0.$$

$$\text{Let } u_x = v_x \sqrt{a + b(-1)^x},$$

$$\text{then } u_{x+1} = v_{x+1} \sqrt{a - b(-1)^x},$$

$$u_{x+2} = v_{x+2} \sqrt{a + b(-1)^x};$$

$$\therefore\ v_{x+2} + \sqrt{a^2 - b^2}\, v_{x+1} + c v_x = 0.$$

4. $$u_{x+1} u_x + a u_{x+1} + b u_x + c = 0.$$

Assume $u_x + a = \dfrac{v_{x+1}}{v_x}$, then $u_{x+1} + a = \dfrac{v_{x+2}}{v_{x+1}}$,

$$\therefore \frac{v_{x+1}}{v_x}\left(\frac{v_{x+2}}{v_{x+1}} - a\right) + b\left(\frac{v_{x+1}}{v_x} - a\right) + c = 0,$$

$$\text{or } v_{x+2} - (a-b)\,v_{x+1} + (c-ab)\,v_x = 0.$$

The two arbitrary constants which will appear in the value of v_x must be reduced to a single constant by the condition of the proposed equation being satisfied.

Ex. $u_x u_{x+1} - 2u_x + 1 = 0$. $u_x = 1 + (x+c)^{-1}$.

5. $u_{x+n} \,.\, u^p{}_{x+n-1} \ldots u_x{}^r = X$.

If we take the logarithm of both sides, and assume $\log u_x = v_x$, we get

$$v_{x+n} + pv_{x+n-1} + qv_{x+n-2} + \ldots + rv_x = \log X;$$

from which we can determine v_x, and then we have $u_x = e^{v_x}$.

Thus, if $u_{x+2} u_{x+1}^{-2} \,.\, u_x = 1$, then $v_{x+2} - 2v_{x+1} + v_x = 0$;

$$\therefore\ v_x = c_0 + c_1 x, \text{ and } u_x = C'\,.\,C^x.$$

Again if $u_{x+2} \,.\, u_{x+1}^{2m} \,.\, u_x^{m2} = a$, then $v_{x+2} + 2mv_{x+1} + m^2 v_x = \log a$;

$$\therefore\ v_x = \frac{\log a}{(1+m)^2} + (c_0 + c_1 x)(-m)^x,$$

which is the value of $\log u_x$.

6. $u^2{}_{x+1} + u_x = 2$.

Assume $u_x = 2\cos v_x$, then $4\cos^2 v_{x+1} + 2\cos v_x = 2$,

$$\therefore\ 2\cos^2 v_{x+1} = 1 - \cos v_x = 2\sin^2 \tfrac{1}{2} v_x,$$

$$\text{or } \cos v_{x+1} = \cos \tfrac{1}{2}(\pi - v_x).$$

$$\therefore\ v_{x+1} + \tfrac{1}{2} v_x = \tfrac{1}{2}\pi, \text{ which gives } v_x = \tfrac{2}{3}\,.\,\tfrac{1}{2}\pi + c\left(-\tfrac{1}{2}\right)^x,$$

$$\therefore\ u_x = 2\cos\left\{\tfrac{1}{3}\pi + c\left(-\tfrac{1}{2}\right)^x\right\}.$$

Similarly, $u_{x+1} - 2u^2_x + 1 = 0$, by assuming $u_x = \cos v_x$, gives $\cos v_{x+1} = \cos 2v_x$, $\therefore\ v_{x+1} - 2v_x = 0$, $v_x = C2^x$, and

$u_x = \cos v_x = \frac{1}{2}\left(e^{v_x\sqrt{-1}} + e^{-v_x\sqrt{-1}}\right) = \frac{1}{2}\left(c^{2^x} + c^{-2^x}\right)$ putting $c = e^{C\sqrt{-1}}$.

7. $u_{x+1}u_x - X(u_{x+1} - u_x) + 1 = 0$.

Let $u_x = \tan v_x$, $\therefore\ X(\tan v_{x+1} - \tan v_x) = 1 + \tan v_{x+1} \tan v_x$,

$$\text{or}\ \tan(v_{x+1} - v_x) = \frac{1}{X}, \quad \therefore\ \Delta v_x = \tan^{-1}\frac{1}{X},$$

$$\text{and}\ u_x = \tan\left(C + \Sigma \tan^{-1}\frac{1}{X}\right).$$

If $X = 1 + x + x^2$, $\qquad u_x = \dfrac{x+c}{1-cx}$.

Similarly $u_{x+2}u_{x+1}u_x = au_x \pm a(u_{x+1} + u_{x+2})$,

by assuming $u_x = \sqrt{a}\,.\tan v_x$, becomes

$$\tan v_x (1 - \tan v_{x+1} \tan v_{x+2}) \pm (\tan v_{x+2} + \tan v_{x+1}) = 0,$$

$$\text{or}\ \tan v_x \pm \tan(v_{x+2} + v_{x+1}) = 0,$$

$\therefore\ v_{x+2} + v_{x+1} \pm v_x = r\pi$, where r is any integer;

$$\therefore\ v_x = \tfrac{1}{3}r\pi + c\cos\frac{2\pi x}{3} + c'\,.\sin\frac{2\pi x}{3},$$

$$\text{or}\ v_x = r\pi + c\left(2\sin\frac{\pi}{10}\right)^x + c'\left(-2\sin\frac{3\pi}{10}\right)^x,$$

according as the upper or lower sign is taken; and then

$$u_x = \sqrt{a}\tan v_x.$$

87. We shall next give some examples of equations which by particular substitutions are reduced to linear equations of the first order.

1. $(ax + b)\Delta^2 u_x + mx\Delta u_x + mnu_x = 0$.

Assume $u_x = \Delta^{n-1}v_x$;

$$\therefore\ (ax + b)\Delta^{n+1}v_x + mx\Delta^n v_x + mn\Delta^{n-1}v_x = 0.$$

But putting $u_x = x - n$ in the formula for transforming $u_{x+n}\Delta^n v_x$ (Art. 34), we have

$$x\Delta^n v_x = \Delta^n \{(x-n)\, v_x\} - n\Delta^{n-1} v_x,$$

$$\therefore\ a\Delta^{n+1}\{(x-n-1)\, v_x\} - a\,(n+1)\,\Delta^n v_x$$
$$+ b\Delta^{n+1} v_x + m\Delta^n \{(x-n)\, v_x\} = 0,$$

$$\text{or } \Delta\,\{a\,(x-n-1)+b\}\, v_x - \{a\,(n+1) - m\,(x-n)\}\, v_x = \Sigma^n 0,$$

$$\therefore\ \{a\,(x-n)+b\}\, v_{x+1} + \{(x-n)\,(m-a) - na - b\}\, v_x = \Sigma^n 0,$$

a linear equation of the first order, which will furnish the value of v_x, and thence the complete integral of the proposed; the supernumerary constants contained in

$$\Sigma^n 0 = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1}$$

being determined by substituting the value of u_x in the proposed. For a particular solution, supposing $\Sigma^n 0 = 0$, and putting

$$A_x = \frac{(a-m)\,(x-n) + na + b}{a\,(x-n)+b},$$

$$\text{we get } v_x = C\,.\,PA_{x-1}, \text{ and } u_x = C\Delta^{n-1}(PA_{x-1}).$$

2. $(ax^2+b)\,\Delta^2 u_x + (r+mx)\,\Delta u_x + (m-n)\,(n+1)\,u_x = 0.$

Let $u_x = \Delta^{n-1} v_x$, then

$$(ax^2+b)\,\Delta^{n+1} v_x + (r+mx)\,\Delta^n v_x + (m-n)\,(n+1)\,\Delta^{n-1} v_x = 0 \ldots (1).$$

But if in the formula for transforming $u_{x+n}\Delta^n v_x$ (Art. 34) we make $u_x = x-n$, $(x-n)^2$, successively, we get values of $x\Delta^n v_x$, $x^2\Delta^{n+1} v_x$; and substituting these values in (1), we find

$$a\Delta^{n+1}\{(x-n-1)^2 v_x\} - a\,(n+1)\,\Delta^n\{(2x-2n-1)\, v_x\}$$
$$+ a\,(n+1)\, n\Delta^{n-1} v_x + b\Delta^{n+1} v_x + r\Delta^n v_x + m\Delta^n\{(x-n)\, v_x\}$$
$$- mn\Delta^{n-1} v_x + (m-n)\,(n+1)\,\Delta^{n-1} v_x = 0,$$
$$\text{or } \Delta^{n+1}\{a\,(x-n-1)^2 + b\}\, v_x$$
$$- \Delta^n\{a\,(n+1)\,(2x-2n-1) - m\,(x-n) - r\}\, v_x$$
$$+ \{(a-1)\,(n+1)\, n + m\}\,\Delta^{n-1} v_x = 0,$$

which is integrable in several cases by the disappearance of one of its terms; thus suppose $a = 1$, $m = 0$, then

$$\Delta\{(x-n-1)^2+b\}\,v_x - \{(n+1)(2x-2n-1)-r\}\,v_x = \Sigma^n 0,$$

$$\text{or}\ \ \{(x-n)^2+b\}\,v_{x+1} - \{x^2-n\,(n+1)+b-r\}\,v_x = \Sigma^n 0,$$

a linear equation of the first order, from which v_x and $u_x = \Delta^{n-1}v_x$ may be determined; and the arbitrary constants contained in $\Sigma^n 0$ must be reduced to the proper number by substituting the value of u_x in the proposed equation. But if we suppose the complementary function $\Sigma^n 0$ to vanish, we may readily obtain two particular values of v_x; for let

$$\frac{x^2-n\,(n+1)+b-r}{(x-n)^2+b} = A_x,$$

$$\text{then}\ \ v_{x+1} - A_x v_x = 0,\ \ v_x = C\,.\,PA_{x-1}.$$

Now replace n by $-(n+1)$ which amounts to assuming $u_x = \Delta^{-n-2}v_x$, then $v_x = C'\,.\,PA'_{x-1}$, where

$$\frac{x^2-n\,(n+1)+b-r}{(x+n+1)^2+b} = A'_x;$$

$$\text{then}\ \ u_x = C\Delta^{n-1}\,(PA_{x-1}) + C'\Sigma^{n+2}\,(PA'_{x-1}),$$

the complete integral of the proposed equation.

Several other examples of the like kind may be seen in the Cambridge and Dublin Mathematical Journal.

87*. The solution of $f(D)\,u_x = b^x$ in the form (Art. 83)

$$u_x = \frac{b^x}{f(b)} + Ca_1^{\,x} + \&c.$$

fails, if b be equal to a root of the auxiliary equation $f(a) = 0$. Suppose that $b = a_1$ a root that occurs twice in $f(a) = 0$; then the value of u_x by changing the constants may be put under the form

$$u_x = \frac{1}{f(b)}\,(b^x - a_1^{\,x} - xha_1^{\,x-1}) + (c_0 + c_1 x)\,a_1^{\,x} + \&c.$$

Now suppose $b = a_1 + h$, where h is very small; then since $f(a_1) = 0$, $f'(a_1) = 0$, we have

$$u_x = \frac{\frac{1}{2}x(x-1)h^2a_1^{x-2} + \&c.}{\frac{1}{2}h^2f''(a_1) + \&c.} + (c_0 + c_1x)a_1^x + \&c.;$$

therefore, making $h = 0$ or $b = a_1$, we get

$$u_x = \frac{x(x-1)}{f''(a_1)}a_1^{x-2} + (c_0 + c_1x)a_1^x + \&c.$$

And if the root to which b becomes equal occur r times, the first term in the value of u_x will be

$$x(x-1)\ldots(x-r+1)\,a_1^{x-r} \div f^{(r)}(a_1).$$

$$\text{Hence in } u_{x+2} - 2a_1u_{x+1} + a_1^2u_x = a_1^x,$$

$$\text{since } f(a) = (a-a_1)^2 \quad f''(a_1) = 2,$$

$$u_x = \tfrac{1}{2}x(x-1)a_1^{x-2} + (c_0 + c_1x)a_1^x;$$

and in examples 3 and 4 of Art. 83 if $\theta = 0$, the first term in the value of u_x becomes

$$\tfrac{1}{2}x(x-1)a^{x-2}, \text{ and } \frac{1}{24}x(x-1)(x-2)(x-3)a^{x-4},$$

respectively.

88. In simultaneous equations, instead of one equation between the independent variable x, and the successive values or differences of the dependent function u_x, we have given two or more equations between x and two or more of its functions u_x, v_x, &c. and their successive values or differences; the object, as before, being to determine each of the unknown functions in terms of x. When the proposed equations are linear with constant coefficients, we may separate the symbols of operation from those of quantity; and then obtain by the ordinary processes of elimination an equation containing only one of the unknown functions; for as the symbols of operation here employed combine according to the same laws as ordinary algebraical quantities, they may be treated precisely as if they were symbols of quantity.

1. $u_{x+2} + cv_{x+1} - c^3 u_x = a^x,$

$v_{x+2} - u_{x+1} - cv_x = a^{-x}.$

These may be written

$$(D^2 - c^3)\, u_x + cDv_x = a^x,$$

$$(D^2 - c)\ v_x - Du_x = a^{-x},$$

which give by the elimination of v_x

$$(D^4 - c^3 D^2 + c^4)\, u_x = (D^2 - c)\, a^x - cDa^{-x};$$

$$\therefore\ u_x = \frac{(a^2 - c)\, a^x}{a^4 - c^3 a^2 + c^4} - \frac{ca^{-x+3}}{1 - c^3 a^2 + c^4 a^4} + \text{complementary function.}$$

Now if c be greater than 2,

$$D^4 - c^3 D^2 + c^4 = (D^2 - \alpha^2)\,(D^2 - \beta^2),$$

and consequently the complementary function

$$= c_1 \alpha^x + c_2 (-\alpha)^x + c_3 \beta^x + c_4 (-\beta)^x.$$

If $c = 2$, the values of α and β become equal to one another; or imaginary if $c < 2$; and the corresponding values of the complementary function may be determined in the usual manner.

The value of u_x being known, we have

$$cv_x = \frac{a^x - (D^2 - c^3)\, u_x}{D} = a^{x-1} - u_{x+1} + c^3 u_{x-1}.$$

2. $u_{x+1} = 2z_x + 12u_x, \quad y_{x+1} = 14z_x,$

$z_{x+1} = y_x + 6z_x + 7u_x.$

Upon introducing the symbol D, and eliminating u_x and y_x, we find $(D - 6)\,(D + 2)\,(D - 14)\, z_x = 0$, which gives

$$z_x = a \,.\, 6^x + b \,.\, (-2)^x + c\,(14)^x;$$

$$\text{then } y_x = 14z_{x-1} = \frac{7a}{3} 6^x - 7b\,(-2)^x + c\,(14)^x,$$

$$u_x = -\frac{a}{3}\, 6^x - \frac{b}{7}(-2)^x + c\,(14)^x.$$

89. It may be observed that the knowledge of the generating function of the second member of a linear equation will sometimes lead to the solution of the equation. For since (Art. 28)

$$G\{\psi(D)\,u_x\} = \psi\left(\frac{1}{t}\right) Gu_x; \qquad \therefore\ Gu_x = \frac{G\{\psi(D)\,u_x\}}{\psi\left(\frac{1}{t}\right)}.$$

Now suppose $\psi(D)\,u_x = X$ to be an equation of Differences, and let $\phi(t)$ be the generating function of X, and therefore of its equal $\psi(D)\,u_x$, then

$$Gu_x = \frac{\phi(t)}{\psi\left(\frac{1}{t}\right)};$$

and if this value of Gu_x can be developed in a series of powers of t, and have w_x for the coefficient of t^x, then the solution of $\psi(D)\,u_x = X$ is $u_x = w_x$ + complementary function.

Thus, for $u_{x+2} - 2\cos\theta u_{x+1} + u_x = \sin x\theta$,

$$\phi(t) = \frac{t\sin\theta}{1 - 2t\cos\theta + t^2} = \ldots + t^x \sin x\theta + \ldots \text{ (Trig. Art. 155)}$$

$$\phi(t) \div \psi\left(\frac{1}{t}\right) = \frac{t^3\sin\theta}{(1 - 2t\cos\theta + t^2)^2} = \ldots$$
$$+ \frac{t^x}{2\sin^2\theta}\{\sin x\theta - x\cos(x-1)\,\theta\sin\theta\} + \ldots$$

the latter expansion being derived from the former by differentiating relative to θ, and reducing;

$$\therefore\ u_x = \frac{1}{2\sin^2\theta}\{\sin x\theta - x\cos(x-1)\,\theta\sin\theta\} + \text{complementary function}$$

$$= -\frac{x}{2\sin\theta}\cos(x-1)\,\theta + c\cos x\theta + c'\sin x\theta.$$

Again, let the proposed equation be $(D-1)^{n-2}u_x = x$, then (Art. 26) $\phi(t) = t(1-t)^{-2}$,

$$\therefore\ \frac{\phi(t)}{\psi\left(\frac{1}{t}\right)} = \frac{t(1-t)^{-2}}{\left(\frac{1}{t} - 1\right)^{n-2}} = t^{n-1}(1-t)^{-n}$$

$$= t^{n-1}\left\{1 + nt + \frac{n(n+1)}{1\,.\,2}t^2 + \ldots + \frac{n(n+1)\ldots x}{1\,.\,2\,.\,3\ldots(x-n+1)}t^{x-n+1} + \ldots\right\};$$

$$\therefore\ w_x = \frac{x(x-1)\ldots n}{1\,.\,2\,.\,3\ldots(x-n+1)} = \frac{x(x-1)\ldots(x-n+2)}{1\,.\,2\,.\,3\ldots(n-1)};$$

$$\therefore\ u_x = \frac{x(x-1)\ldots(x-n+2)}{1\,.\,2\,.\,3\ldots(n-1)} + \alpha + \beta x + \ldots + \lambda x^{n-3},$$

which agrees with the result obtained by direct integration of $\Delta^{n-2} u_x = x$.

Equations of Partial Differences and Mixed Differences.

90. If in $u_{x,y}$ a function of two independent variables x and y, x be changed into $x+h$, and y into $y+k$, the resulting value $u_{x+h,\,y+k}$ is called the first total successive value of $u_{x,y}$, and is denoted by $Du_{x,y}$, so that $Du_{x,y} = u_{x+h,\,y+k}$; consequently

$$D^2 u_{x,y} = u_{x+2h,\,y+2k},\ \&\text{c.}\ \ D^n u_{x,y} = u_{x+nh,\,y+nk};$$

and as $u_{x+h,\,y+k} = e^{\frac{d}{dx}+\frac{d}{dy}} u_{x,y}$, D is here equivalent to $e^{\frac{d}{dx}+\frac{d}{dy}}$. Also if from the altered value of the function we subtract the original value, the result is called the first total difference of $u_{x,y}$, and is denoted by $\Delta u_{x,y}$, so that

$$\Delta u_{x,y} = u_{x+h,\,y+k} - u_{x,y} = Du_{x,y} - u_{x,y} = (D-1)\,u_{x,y}.$$

Consequently we have, as in the case of a function of one variable, Δ equivalent to $D-1$; and $1+\Delta$ equivalent to D; hence we must have

$$\Delta^2 u_{x,y} = (D-1)^2\,u_{x,y},\quad D^2 u_{x,y} = (1+\Delta)^2\,u_{x,y},\ \&\text{c.};$$

and generally

$$\Delta^n u_{x,y} = (D-1)^n\,u_{x,y},\quad D^n u_{x,y} = (1+\Delta)^n\,u_{x,y};$$

which become by development

$$\Delta^n u_{x,y} = D^n u_{x,y} - nD^{n-1} u_{x,y} + \frac{n(n-1)}{1\,.\,2} D^{n-2} u_{x,y} - \&\text{c.}$$

$$= u_{x+nh,\,y+nk} - n u_{x+(n-1)h,\,y+(n-1)k} + \&\text{c.},$$

$$D^n u_{x,y} = u_{x,y} + n\Delta u_{x,y} + \frac{n(n-1)}{1\,.\,2}\,\Delta^2 u_{x,y} + \&\text{c.};$$

where Δ, D express total operations that refer to both of the independent variables.

If there be given $\Delta u_{x,y} = f(x, y)$, a rational and integral function of x and y, we may find the value of $u_{x,y}$ by the method of Art. 44; thus if $\Delta u_{x,y} = x + y + 1$, then

$$u_{x,y} = \tfrac{1}{2}(x^2 + y^2) + C_{x-y},$$

where C_{x-y} denotes any function of $x - y$, the increments of both variables being taken equal to unity.

91. Again, if in $u_{x,y}$ we change x into $x + h$ without altering y, the result $u_{x+h,y}$ is called the first partial successive value relative to x, and is denoted by $D_x u_{x,y}$, so that $D_x u_{x,y} = u_{x+h,y}$; consequently

$$D_x^2 u_{x,y} = u_{x+2h,y}, \text{ \&c., } D_x^n u_{x,y} = u_{x+nh,y};$$

and as $u_{x+h,y} = e^{h\frac{d}{dx}} u_{x,y}$, D_x is equivalent to $e^{h\frac{d}{dx}}$. Exactly in the same way we have, relative to the other variable y,

$$D_y u_{x,y} = u_{x,y+k}, \quad D_y^n u_{x,y} = u_{x,y+nk},$$

D_y being equivalent to $e^{k\frac{d}{dy}}$; hence $D = D_x . D_y$. Also the differences $u_{x+h,y} - u_{x,y}$, $u_{x,y+k} - u_{x,y}$, are called the partial differences of $u_{x,y}$ relative to x and y respectively, and are denoted by Δ_x, Δ_y; so that

$$\Delta_x u_{x,y} = u_{x+h,y} - u_{x,y} = (D_x - 1)\, u_{x,y};$$

$$\Delta_y u_{x,y} = u_{x,y+k} - u_{x,y} = (D_y - 1)\, u_{x,y};$$

and generally

$$\Delta_x^n u_{x,y} = (D_x - 1)^n u_{x,y}, \quad \Delta_y^n = (D_y - 1)^n u_{x,y};$$

and the symbols Σ_x, Σ_y, which are equivalent to Δ_x^{-1}, Δ_y^{-1}, are used to denote the operation of integrating $u_{x,y}$ with respect to the variable x only, and the variable y only, respectively.

In the investigations which follow, the values of h and k will be taken equal to unity, unless the contrary be stated.

92. By Equations of Partial Differences are meant relations between a function of two independent variables $u_{x,y}$, some of its partial successive values or differences, and the variables x and y. As the symbols of operation here employed combine in subjection to the same fundamental laws as algebraical quantities, we may, by separating the symbols, resolve these equations when linear or otherwise, by the same processes as ordinary equations of differences of the like class: but we shall first employ an elementary method.

Ex. 1. $u_{x+n,y} + p_1 u_{x+n-1,y+1} + \dots + p_n u_{x,y+n} = 0$ (1), the coefficients being constant, and the sum of the subscribed indices the same in every term. If the second member be q instead of zero, it is easily seen that the value of $u_{x,y}$ determined from (1) would have to be increased by $\dfrac{q}{f(1)}$.

Assume $u_{x,y} = a^x . C_{x+y}$, where C_{x+y} is any function of the binomial $x+y$, then the first member of (1) becomes

$$a^x (a^n + p_1 a^{n-1} + \dots + p_n) C_{x+y+n}$$

which is reduced to zero if a be taken equal to any one of the roots real or imaginary, a_1, a_2, ... a_n of the auxiliary equation

$$f(a) = a^n + p_1 a^{n-1} + \dots + p_n = 0.$$

Hence, if a_1, a_2, &c., be all unequal, the complete value of $u_{x,y}$ involving n arbitrary functions is

$$u_{x,y} = a_1^x C'_{x+y} + a_2^x C''_{x+y} + \dots + a_n^x C^{(n)}_{x+y},$$

which shews that arbitrary functions of the binomial $x+y$ here stand in the places of the arbitrary constants that enter into the solutions of ordinary equations of Differences. But if $a_2 = a_1$, first suppose $a_2 = a_1 + h$ where h is very small, then

$$a_1^x C'_{x+y} + a_2^x C''_{x+y} = a_1^x (C'_{x+y} + C''_{x+y})$$

$$+ \frac{h}{a_1} C''_{x+y} . x a_1^x \{1 + \tfrac{1}{2}(x-1) \frac{h}{a_1} + \text{\&c.}\},$$

$$= a_1^x \cdot c'_{x+y} + c''_{x+y} \cdot x a_1^x \{1 + \tfrac{1}{2}(x-1)\frac{h}{a_1} + \&c.\},$$

replacing the arbitrary functions

$$C'_{x+y} + C''_{x+y} \text{ and } \frac{h}{a_1} C''_{x+y}$$

by other arbitrary functions c'_{x+y} and c''_{x+y}; now make $h = 0$, then the equal roots a_1, a_2 produce in the value of $u_{x,y}$ the terms $a_1^x (c'_{x+y} + x c''_{x+y})$. If a_1, a_2 be a pair of imaginary roots and

$$= \rho (\cos\theta \pm \sqrt{-1} \sin\theta),$$

then it is easily seen that $a_1^x C'_{x+y} + a_2^x C''_{x+y}$ takes the form

$$\rho^x (c_{x+y} \cos x\theta + c'_{x+y} \sin x\theta).$$

It may be observed that the assumption $u_{x,y} = a^x . C_{x+y}$ is symmetrical with respect to x and y, for it may be replaced by

$$u_{x,y} = \left(\frac{1}{a}\right)^y . a^{x+y} . C_{x+y} = b^y . C'_{x+y} .$$

Ex. 1. $u_{x+1,y} - a u_{x,y+1} = b$. Here $u_{x,y} = a^x . C_{x+y} + \dfrac{b}{1-a}$.

Ex. 2. $u_{x+2,y} - 2a u_{x+1,y+1} + a^2 u_{x,y+2} = b$;

$$u_{x,y} = a^x (c_{x+y} + x . c'_{x+y}) + \frac{b}{(1-a)^2}.$$

Ex. 3. $u_{x+2,y} - 2a \cos\theta\, u_{x+1,y+1} + a^2 u_{x,y+2} = b.$

$$u_{x,y} = a^x (c_{x+y} \cos x\theta + c'_{x+y} \sin x\theta) + \frac{b}{1 - 2a\cos\theta + a^2}.$$

93. The preceding results can of course be readily obtained by separating the symbols; but that method may be applied with still greater advantage to the case where the second member is V a function of x and y. For the proposed equation may be written

$$\{(D_x D_y^{-1})^n + p_1 (D_x D_y^{-1})^{n-1} + \ldots + p_n\} D_y^n u_{x,y} = V,$$

instead of $u_{x+n,y} + p_1 u_{x+n-1,y+1} + \ldots + p_n u_{x,y+n} = V.$

Let a be a root that occurs singly, and b a root that occurs m times, in the auxiliary equation $z^n + p_1 z^{n-1} + \ldots + p_n = 0$: then by resolution into partial fractions we shall have

$$\frac{1}{z^n + p_1 z^{n-1} + \ldots + p_n} = \frac{A}{z-a} + \frac{B_m}{(z-b)^m} + \frac{B_{m-1}}{(z-b)^{m-1}} + \ldots + \frac{B_1}{z-b} + \&c.,$$

where A, B_m, &c. are known in terms of p_1, p_2, &c.; hence substituting $D_x D_y^{-1}$ for z, we get

$$u_{x,y} = \left\{ \frac{AD_y}{D_x - aD_y} + \frac{B_m D_y^{\ m}}{(D_x - bD_y)^m} + \frac{B_{m-1} D_y^{\ m-1}}{(D_x - bD_y)^{m-1}} + \ldots \right.$$

$$\left. + \frac{B_1 D_y}{D_x - bD_y} \right\} D_y^{\ -n} V;$$

but $A\ (D_x - aD_y)^{-1} D_y^{\ -n+1} V = A\ (aD_y)^{x-1} \Sigma_x\ (aD_y)^{-x} D_y^{\ -n+1} V$; &c.;

since the operations denoted by D_x, D_y, are independent of one another:

$$\therefore\ u_{x,y} = A\ (aD_y)^{x-1}\ \Sigma_x\ (aD_y)^{-x}\ D_y^{\ -n+1} V$$

$$+ B_m (bD_y)^{x-m}\ \Sigma_x^{\ m}\ (bD_y)^{-x}\ D_y^{\ -n+m} V + \&c.$$

Instead of reserving the complementary functions under the sign of integration, we may obtain them explicitly by supposing $V = 0$; then since

$$\Sigma_x 0 = c_y,\ \ \Sigma_x^{\ m} 0 = c^0_{\ y} + xc'_{\ y} + x^2 c''_{\ y} + \ldots + x^{m-1} c_y^{\ (m-1)},$$

$$\text{and}\ \ D_y^{\ x-1} c_y = c_{x+y-1},\ \ D_y^{\ x-m}\ c_y = c_{x+y-m},$$

the part of the complementary function due to the root a and the m roots b will be, changing the arbitrary functions,

$$Aa^x c_{x+y} + B_m b^x \{c^0_{\ x+y} + xc'_{\ x+y} + \ldots + x^{m-1} c^{(m-1)}_{x+y}\},$$

and similar terms will be introduced by the other roots of the auxiliary equation.

If any of the roots a, b, &c. be imaginary, these results may be transformed into real expressions, as shewn above.

Ex. 1. $u_{x+1,y} - u_{x,y+1} = xy$,

or $(D_x - D_y)\, u_{x,y} = xy$, $\therefore\ u_{x,y} = (D_x - D_y)^{-1} xy = (D_y)^{x-1} \Sigma_x D_y^{-x}\, xy$;

$$\text{but } D_y^{-x} xy = x\,(y-x) = x\,(y+1) - x\,(x+1);$$

$$\therefore\ u_{x,y} = D_y^{x-1} \left\{ \frac{(x-1)\,x\,(y+1)}{2} - \frac{(x-1)\,x\,(x+1)}{3} + C_y \right\}$$

$$= \tfrac{1}{6}\,(x-1)\,x\,(x+3y-2) + C_{x+y}.$$

Ex. 2. $u_{x+2,y} - 2u_{x+1,y+1} + u_{x,y+2} = x + y$,

$$\text{or } (D_x - D_y)^2\, u_{x,y} = x + y;$$

$$\therefore\ u_{x,y} = (D_x - D_y)^{-2}\,(x+y) = D_y^{x-2} \Sigma_x^{\,2}\, D_y^{-x}\,(x+y)$$

$$= D_y^{x-2} \Sigma_x^{\,2} y = D_y^{x-2} \left\{ \tfrac{1}{2}\,(x-1)\,xy + xC_y + C'_y \right\}$$

$$= \tfrac{1}{2}\,(x-1)\,x\,(y+x-2) + xc_{x+y} + c'_{x+y}.$$

Ex. 3. $u_{x+3,y} - 12u_{x+1,y+2} + 16u_{x,y+3} = c^{x+y}$,

$$\text{or } (D_x - 2D_y)^2\,(D_x + 4D_y)\, u_{x,y} = c^{x+y};$$

and since D_x, D_y, applied to c^{x+y} are each equivalent to multiplying it by c,

$$\therefore\ u_{x,y} = \frac{c^{x+y}}{(-c)^2 \,.\, 5c} + \&\text{c.} = \tfrac{1}{5} c^{x+y-3} + 2^x\,(c_{x+y} + xc'_{x+y}) + (-4)^x c''_{x+y}.$$

Ex. 4. $u_{x+2,y} - 2a\cos\theta\, u_{x+1,y+1} + a^2 u_{x,y+2} = V.$

$$u_{x,y} = a^{x-2}\,\frac{\sin\,(x-1)\,\theta}{\sin\theta}\, D_y^{x-1}\, \Sigma_x\, (a^{-x}\cos x\theta\, D_y^{-x-1} V)$$

$$- a^{x-2}\,\frac{\cos\,(x-1)\,\theta}{\sin\theta}\, D_y^{x-1} \Sigma_x (a^{-x}\sin x\theta\, D_y^{-x-1} V)$$

$$+ a^x\,(\cos x\theta \,.\, c_{x+y} + \sin x\theta \,.\, c'_{x+y}).$$

Ex. 5. $(D_x - a)\,(D_y - b)\, u_{x,y} = V$,

$$\text{or } u_{x+1,y+1} - au_{x,y+1} - bu_{x+1,y} + abu_{x,y} = V;$$

$$\therefore\ (D_y - b)\, u_{x,y} = (D_x - a)^{-1} V = a^{x-1} \Sigma_x a^{-x} V + a^{x-1} C_y,$$

$$u_{x,y} = a^{x-1} b^{y-1}\, \Sigma_y\, (b^{-y} \Sigma_x a^{-x} V) + a^{x-1} b^{y-1} C'_x + a^{x-1} C_y$$

$$= a^{x-1} b^{y-1}\, \Sigma_y\, (b^{-y}\, \Sigma_x a^{-x} V) + b^y c'_x + a^x c''_y.$$

If $V = c^{x-y}$, since D_x, D_y are in that case equivalent respectively to c and c^{-1};

$$u_{x,y} = \frac{c^{x-y}}{(c-a)(c^{-1}-b)} + b^y c'_x + a^x c''_y.$$

Equations of Mixed Differences.

94. Equations of Mixed Differences are those in which the differential coefficients of the dependent variable u_x, as well as its differences or successive values, are involved along with the principal variable x. When the equation is linear with respect to the successive values and their differential coefficients, and the coefficients of the several terms are constant, solutions may be obtained by proceeding in the same way as for linear equations of differences of the like kind. It is plain that any differential equation

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \text{\&c.}\right) = 0,$$

which admits of a solution $y = X$, will lead by substituting

$$y = u_{x+n} + p_1 u_{x+n-1} + \dots + p_n u_x,$$

where p_1, p_2, &c. are constants, to a linear equation of differences of the kind that has been solved.

1. $$\frac{d}{dx}(u_{x+n} + p_1 u_{x+n-1} + \dots + p_n u_x)$$
$$+ m(u_{x+n} + p_1 u_{x+n-1} + \dots + p_n u_x) = X.$$

Multiplying both sides by e^{mx} and integrating, we get

$$u_{x+n} + p_1 u_{x+n-1} + \dots + p_n u_x = e^{-mx} \int dx e^{mx} X,$$

an ordinary linear equation of differences. Thus suppose the proposed equation to be

$$\frac{du_{x+1}}{dx} + a\frac{du_x}{dx} + bu_{x+1} + abu_x = 0,$$

then $u_x = Cm^x + C'(-a)^x$, where $m = e^{-b}$.

2. $\frac{d}{dx}(u_{x+1} - 2u_x) + 3u_{x+1} - 6u_x = e^{mx}$,

$$\text{then } u_{x+1} - 2u_x = \frac{e^{mx}}{m+3} + C_0 e^{-3x};$$

$$\therefore\ u_x = \frac{e^{mx}}{(m+3)(e^m - 2)} + C \,.\, e^{-3x} + C' \,.\, 2^x.$$

3. $(1 + ae^{x+1})\, u_{x+1} - b\,(1 + ae^x)\, \frac{d}{dx} u_x + bu_x = c.$

Assume $$u_x(1 + ae^x) = v_x;$$

$$\therefore\ ae^x u_x + (1 + ae^x)\frac{d}{dx} u_x - \frac{dv_x}{dx} = 0,$$

and multiplying this by b and adding it to the proposed, we get

$$v_{x+1} - b\frac{dv_x}{dx} + bv_x = c.$$

Now, let $v_x = e^{mx} + k$;

$$\therefore\ e^{m(x+1)} + k - mb \,.\, e^{mx} + be^{mx} + bk = c:$$

let $k + bk = c$, then $e^m - (m-1)\, b = 0$; and if μ be a value of m which satisfies this equation, then a particular value of u_x is

$$\frac{1}{1 + ae^x}\left(Ce^{\mu x} + \frac{c}{1+b}\right).$$

95. The following are examples of Equations of Mixed Partial Differences.

1. $u_{x+1,y} - \frac{d}{dy} u_{x,y} = c$,

$$\text{or } D_x u_{x\,y} - \frac{d}{dy} u_{x,y} = c,$$

$$\therefore\ u_{x,y} = \left(D_x - \frac{d}{dy}\right)^{-1} c = \left(\frac{d}{dy}\right)^{x-1} \left\{\Sigma_x \left(\frac{d}{dy}\right)^{-x} c + \frac{d}{dy}\, C_y\right\},$$

putting $\frac{d}{dy} C_y$ for an arbitrary function of y;

$$\therefore\ u_{x,y} = c + \left(\frac{d}{dy}\right)^x C_y;$$

and if $c = 0$, we have for the equation

$$u_{x+1,y} - \frac{d}{dy} u_{x,y} = 0, \quad u_{x,y} = \left(\frac{d}{dy}\right)^x C_y.$$

2. $$u_{x+1,y} - a\frac{d}{dy} u_{x,y} = V;$$

$$\therefore\ u_{x,y} = \left(D_x - a\frac{d}{dy}\right)^{-1} V = \left(a\frac{d}{dy}\right)^{x-1} \Sigma_x \left(a\frac{d}{dy}\right)^{-x} V.$$

Suppose $V = xe^y$, then the effect of any power, positive or negative, of $\frac{d}{dy}$ upon V, is to multiply it by 1;

$$\therefore\ \Sigma_x \left(a\frac{d}{dy}\right)^{-x} V = e^y \Sigma_x a^{-x} x = \frac{e^y}{a^{x-1}} \frac{x(1-a)-1}{(1-a)^2} + a\frac{d}{dy} C_y,$$

$$\therefore\ u_{x,y} = e^y \frac{x(1-a)-1}{(1-a)^2} + a^x \left(\frac{d}{dy}\right)^x C_y.$$

3. $$u_{x+2,y} - 2a\frac{d}{dy} u_{x+1,y} + a^2 \left(\frac{d}{dy}\right)^2 u_{x,y} = V;$$

$$\therefore\ u_{x,y} = \left(D_x - a\frac{d}{dy}\right)^{-2} V = \left(a\frac{d}{dy}\right)^{x-2} \Sigma_x^2 \left(a\frac{d}{dy}\right)^{-x} V.$$

Let $V = xe^y$, then proceeding as in the last example,

$$u_{x,y} = a^{x-2} e^y \Sigma^2 (a^{-x} x) = e^y \frac{x(1-a)-2}{(1-a)^3} + a^x \left(\frac{d}{dy}\right)^x C_y + xa^x \left(\frac{d}{dy}\right)^x C_y^1.$$

4. $$\Delta_x u_{x,y} - a\frac{d}{dy} u_{x,y} = c.$$

As this may be written

$$\Delta_x (u_{x,y} - cx) - a\frac{d}{dy}(u_{x,y} - cx) = 0,$$

we may integrate it supposing $c=0$, and then add cx to the result. Assume $u_{x,y}=a^x e^{-\frac{y}{a}} v_{x,y}$;

$$\therefore\ a^x e^{-\frac{y}{a}} (av_{x+1,y}-v_{x,y}) - a^x e^{-\frac{y}{a}} \left(-v_{x,y}+a\frac{d}{dy}v_{x,y}\right)=0;$$

$$\text{or } v_{x+1,y}-\frac{d}{dy}v_{x,y}=0; \quad \therefore\ v_{x,y}=\left(\frac{d}{dy}\right)^x C_y. \quad \text{(Ex. 1)};$$

$$\therefore\ u_{x,y}=cx+a^x e^{-\frac{y}{a}}\left(\frac{d}{dx}\right)^x C_y.$$

Continued Fractions.—Functional Equations.

96. The determination of the values of continued fractions in certain cases gives rise to equations of differences which can be integrated as linear equations.

1. To determine the value of the continued fraction,

$$u_x=\frac{c}{a+}\,\frac{c}{a+\dots}\,\frac{c}{a} \text{ to } x \text{ fractional terms};$$

$$\therefore\ u_{x+1}=\frac{c}{a+u_x}, \text{ or } u_{x+1}(u_x+a)=c;$$

Assume $u_x+a=\dfrac{v_{x+1}}{v_x}$, then $v_{x+2}-av_{x+1}-cv_x=0$;

$$\therefore\ v_x=c_1\alpha^x+c_2\beta^x,\ \alpha \text{ and } \beta \text{ being roots of } k^2-ak-c=0;$$

$$\therefore\ u_x=\frac{c_1\alpha^{x+1}+c_2\beta^{x+1}}{c_1\alpha^x+c_2\beta^x}-a=\frac{c_1\alpha^x(\alpha-a)+c_2\beta^x(\beta-a)}{c_1\alpha^x+c_2\beta^x}$$

$$=-\alpha\beta\,\frac{c_1\alpha^{x-1}+c_2\beta^{x-1}}{c_1\alpha^x+c_2\beta^x}, \text{ since } \alpha+\beta=a;$$

$$\text{or } u_x=c\,.\,\frac{\alpha^{x-1}+c_3\beta^{x-1}}{\alpha^x+c_3\beta^x}, \text{ putting } c_3=\frac{c_2}{c_1};$$

$$\therefore\ u_1=\frac{c}{a}=c\,\frac{1+c_3}{\alpha+c_3\beta}, \quad \therefore\ c_3=-\frac{\beta}{\alpha},$$

$$\therefore\ u_x=c\,\frac{\alpha^x-\beta^x}{\alpha^{x+1}-\beta^{x+1}}.$$

If α and β be imaginary, so that

$$k^2 - ak - c = k^2 - 2k\rho \cos\theta + \rho^2,$$

$$\text{then } u_x = -\rho \frac{\sin x\theta}{\sin(x+1)\theta}.$$

2. To find the value of the continued fraction

$$u_x = \frac{c}{a+} \frac{c}{a+\ldots} \frac{c}{a+} \frac{c}{b},$$

to x fractional terms, the last term being $\frac{c}{b}$.

Proceeding as in the last example, we find

$$u_x = c \, . \frac{\alpha^{x-1} + c_3\beta^{x-1}}{\alpha^x + c_3\beta^x},$$

$$\text{and } u_1 = \frac{c}{b} = c \, . \frac{1 + c_3}{\alpha + c_3\beta}, \quad \therefore \ c_3 = -\frac{b-\alpha}{b-\beta},$$

$$\therefore \ u_x = c \, . \frac{b(\alpha^{x-1} - \beta^{x-1}) + c(\alpha^{x-2} - \beta^{x-2})}{b(\alpha^x - \beta^x) + c(\alpha^{x-1} - \beta^{x-1})}.$$

If $c = -1$, $a = 2\cos\theta$, $b = \cos\theta$, then it will be found that

$$u_x = -\frac{\cos(x-1)\theta}{\cos x\theta}.$$

3. To find the value of the continued fraction to x fractional terms,

$$u_x = \frac{1}{a+} \frac{1}{b+} \frac{1}{a+} \frac{1}{b+\ldots}.$$

$$\text{Here } u_{x+2} = \cfrac{1}{a + \cfrac{1}{b + u_x}}, \quad \text{or} \quad \frac{u_{x+2}}{1 - au_{x+2}} = b + u_x.$$

$$\text{Assume } 1 - au_x = \frac{v_{x-1}}{v_{x+1}}, \quad \therefore \ 1 - au_{x+2} = \frac{v_{x+1}}{v_{x+3}},$$

$$\therefore \ \frac{1}{a}\left(1 - \frac{v_{x+1}}{v_{x+3}}\right)\frac{v_{x+3}}{v_{x+1}} = b + \frac{1}{a}\left(1 - \frac{v_{x-1}}{v_{x+1}}\right),$$

$$\therefore\ v_{x+3} - v_{x+1} = abv_{x+1} + v_{x+1} - v_{x-1},$$

$$\text{or}\ v_{x+3} - (2+ab)\,v_{x+1} + v_{x-1} = 0,$$

of which the auxiliary equation is

$$k^4 - (2+ab)\,k^2 + 1 = (k^2 - \sqrt{ab}\,k - 1)\,(k^2 + \sqrt{ab}\,k - 1) = 0.$$

Let $\alpha,\ -\dfrac{1}{\alpha},\ -\alpha,\ \dfrac{1}{\alpha}$, be the roots of this equation, then

$$v_x = a^x\,\{c_1 + c_2\,(-1)^x\} + \frac{1}{a^x}\,\{c_3 + c_4\,(-1)^x\},$$

$$\therefore\ au_x = 1 - \frac{\alpha^{x-1} + p_x\alpha^{-(x-1)}}{\alpha^{x+1} + p_x\alpha^{-(x+1)}},$$

$$\text{where}\ p_x = \frac{c_3 + c_4\,(-1)^{x-1}}{c_1 + c_2\,(-1)^{x-1}} = C + C'\,(-1)^{x-1};$$

$$\therefore\ au_1 = 1 = 1 - \frac{1+p_1}{\alpha^2 + p_1\alpha^{-2}},\quad \therefore\ p_1 = -1 = C + C',$$

$$au_2 = \frac{ab}{1+ab} = 1 - \frac{\alpha + p_2\alpha^{-1}}{\alpha^3 + p_2\alpha^{-3}},$$

$$\therefore\ ab + 1 = \alpha^2 - 1 + \alpha^{-2} = \frac{\alpha^3 + p_2\alpha^{-3}}{\alpha + p_2\alpha^{-1}},\quad \because\ \alpha - \frac{1}{\alpha} = \sqrt{ab};$$

$$\therefore\ p_2 = 1 = C - C',\quad \therefore\ C = 0,\quad C' = -1,\ \text{and}\ p_x = (-1)^x,$$

$$\therefore\ au_x = 1 - \frac{\alpha^{x-1} + (-1)^x\alpha^{-x+1}}{\alpha^{x+1} + (-1)^x\alpha^{-x-1}}.$$

4. Suppose b to be essentially negative and $= -c$, so that the continued fraction is

$$\frac{1}{a+}\ \frac{1}{-c+}\ \frac{1}{a+}\ \frac{1}{-c+\ldots},$$

and let $\sqrt{ab} = \sqrt{-1}\,\sqrt{ac} = 2\sqrt{-1}\,\sin\theta$, then

$$k^2 - 2\sqrt{-1}\,\sin\theta\,k - 1 = 0,\ \text{and}\ \alpha = \cos\theta + \sqrt{-1}\,\sin\theta;$$

therefore, according as x is odd, or even,

$$au_x = 1 - \frac{\sin(x-1)\theta}{\sin(x+1)\theta}, \text{ or } 1 - \frac{\cos(x-1)\theta}{\cos(x+1)\theta}.$$

Functional Equations.

97. If a function $\phi(x)$ is of such a nature that when it is twice performed on a quantity, the result is the quantity itself, that is $\phi\{\phi(x)\}$ or $\phi^2(x) = x$, then it is called a periodic function of the second order; and if $\phi(x)$ be such that $\phi^n(x) = x$, then $\phi(x)$ is termed a periodic function of the n^{th} order. Any symmetrical function of x and y, put equal to zero, will by resolution give for the value of y a function of x, which is a periodic function of the second order. Consequently,

$$\frac{a}{x}, \quad a - x, \quad \frac{x}{x-1}, \quad \frac{1-x}{1+x}, \quad \sqrt{1-x^2},$$

are all periodic functions of the second order. Examples of periodic functions of the third order are $\dfrac{1}{1-x}$, $\sqrt{1-x^{-2}}$; and of the fourth order $\frac{1}{2}\dfrac{1}{1-x}$, $\dfrac{2}{2-x}$. Functional equations involving these expressions may be frequently solved by a simple elimination; as will be seen in the following instances.

Ex. 1.
$$\{f(x)\}^2 . f\left(\frac{1-x}{1+x}\right) = c^2 x;$$

here, since $\dfrac{1-x}{1+x}$ is a periodic function of the second order, replacing x by $\dfrac{1-x}{1+x}$, we get

$$\left\{f\left(\frac{1-x}{1+x}\right)\right\}^2 f(x) = c^2 . \frac{1-x}{1+x};$$

hence dividing the square of the proposed equation by this, we have

$$\{f(x)\}^3 = \frac{1+x}{1-x} c^2 x^2, \text{ or } f(x) = \left(\frac{1+x}{1-x} c^2 x^2\right)^{\frac{1}{3}}.$$

Ex. 2.
$$f(x) + a . f\left(\frac{1}{1-x}\right) = \frac{1}{x};$$

here since $\frac{1}{1-x}$ is a periodic function of the third order, replacing x twice in succession by $\frac{1}{1-x}$, we get

$$f\left(\frac{1}{1-x}\right)+af\left(1-\frac{1}{x}\right)=1-x,$$

$$f\left(1-\frac{1}{x}\right)+af(x)=\frac{x}{x-1}.$$

Now multiply the second of these equations by $-a$, and the third by a^2, and we find by adding them together,

$$(1+a^3)f(x)=\frac{1}{x}+a(x-1)+\frac{a^2x}{x-1}.$$

98. Other functional equations may be sometimes solved by the aid of the Calculus of Finite Differences, or by particular artifices.

Ex. 1. $$f(x)+f(a-x)=c^2.$$

Assume $$x=u_z, \quad a-x=u_{z+1};$$

$$\therefore \; u_{z+1}+u_z=a, \quad \text{or } u_z=C(-1)^z+\tfrac{1}{2}a.$$

But $f(u_{z+1})+f(u_z)=c^2$, which may be written

$$v_{z+1}+v_z=c^2;$$

$$\therefore \; v_z=C'(-1)^z+\tfrac{1}{2}c^2=\frac{C'}{C}(u_z-\tfrac{1}{2}a)+\tfrac{1}{2}c^2=f(u_z);$$

$$\therefore \; f(x)=m(x-\tfrac{1}{2}a)+\tfrac{1}{2}c^2,$$

m being an arbitrary constant.

In the same way we may find a particular solution of the equation

$$f(a+x).f(a-x)=a^2-x^2;$$

or we may use for that purpose the following artifice,

$$\frac{f(a+x)}{a+x}\cdot\frac{f(a-x)}{a-x}=1, \quad \text{or } \phi(x).\phi(-x)=1,$$

which is satisfied by $\phi(x) = C^x$;

$$\therefore f(a+x) = (a+x)\, C^x = (a+x)\, C^{(a+x)-a};$$

$$\therefore f(x) = x\, C^{x-a}.$$

Ex. 2. $$f(bx) = nf(x).$$

Let $$bx = u_{z+1}, \quad x = u_z;$$

$$\therefore u_{z+1} - bu_z = 0, \quad \text{or} \quad u_z = \frac{1}{c} b^z = x;$$

$$\text{but } f(u_{z+1}) - nf(u_z) = 0, \quad \text{or} \quad v_{z+1} - nv_z = 0;$$

$$\therefore v_z = n^z . \phi(\cos 2\pi z) = n^{\frac{\log cx}{\log b}} \times \phi\left\{\cos\left(\frac{2\pi}{\log b} \log cx\right)\right\} \text{ (Art. 41)},$$

$$\text{but } v_z = f(u_z) = f(x),$$

$$\therefore f(x) = (cx)^{\frac{\log n}{\log b}} \times \phi\left\{\cos\left(\frac{2\pi \log cx}{\log b}\right)\right\}.$$

Ex. 3. $$f(a+bx) = nf(x).$$

Assume $a + bx = u_{z+1}$, $x = u_z$; then proceeding as in the last Example we have

$$f(x) = \left(cx - \frac{ca}{1-b}\right)^{\frac{\log n}{\log b}} \times \phi\left\{\cos \frac{2\pi}{\log b} \log\left(cx - \frac{ca}{1-b}\right)\right\};$$

and for a particular value, when $c = 1$, and $\phi(\cos 2\pi z) = 1$,

$$f(x) = \left(x - \frac{a}{1-b}\right)^{\frac{\log n}{\log b}}.$$

Ex. 4. $$f(x) . f(y) = f(x+y) + f(x-y).$$

Expanding the second member by Taylor's theorem and dividing both sides by $f(x)$, we get

$$f(y) = 2\left\{1 + \tfrac{1}{2} \frac{y^2}{f(x)} \cdot \frac{d^2 f(x)}{dx^2} + \frac{1}{1.2.3.4} \cdot \frac{y^4}{f(x)} \cdot \frac{d^4 f(x)}{dx^4} + \&c.\right\}.$$

But as $f(y)$ does not involve x, the coefficients of y^2, y^4, &c. in the second member must be constants; let b be the value of the coefficient of y^2, then

$$\frac{d^2 f(x)}{dx^2} = b \,.f(x),$$

$$\therefore \frac{d^4 f(x, y)}{dx^4} = b \,. \frac{d^2 f(x)}{dx^2} = b^2.f(x), \text{ \&c.};$$

$$\therefore f(y) = 2\left(1 + \tfrac{1}{2} by^2 + \frac{1}{1.2.3.4} b^2 y^4 + \text{\&c.}\right),$$

or replacing b by another constant $-c^2$,

$$f(y) = 2\left(1 - \tfrac{1}{2} c^2 y^2 + \frac{1}{1.2.3.4} c^4 y^4 - \text{\&c.}\right) = 2 \cos cy.$$

99. We shall conclude this Section by two or three Problems, the solution of which is effected by equations of Differences.

1. To find the value of $u_x = \sqrt{2 - \sqrt{2 - \sqrt{2 \ldots}}}$ to x terms.

Since $u_{x+1} = \sqrt{2 - u_x}$, $u^2_{x+1} + u_x = 2$;

of which the integral is $u_x = 2 \cos \{c\,(-\tfrac{1}{2})^x + \tfrac{1}{3}\pi\}$; (Art. 86)

$$\therefore u_1 = 2 \cos\left(-\frac{c}{2} + \frac{\pi}{3}\right) = \sqrt{2} = 2 \cos \frac{\pi}{4}, \quad \therefore c = \frac{\pi}{6};$$

$$\therefore u_x = 2 \cos \left\{\frac{\pi}{6}\,(-\tfrac{1}{2})^x + \frac{\pi}{3}\right\} = 2 \sin \frac{\pi}{6}\,\{1 - (-\tfrac{1}{2})^x\}.$$

2. A swan breeds three cygnets in its second year, and four every succeeding year; and the young ones all breed according to the same law; required the number at the end of the x^{th} year.

Let $u_x =$ number at end of x^{th} year;
then in $(x+1)^{\text{th}}$ year, there will be bred three each by those in their second year, and four each by all the others;

$$\therefore u_{x+1} = u_x + 3\,(u_{x-1} - u_{x-2}) + 4u_{x-2} = u_x + 3u_{x-1} + u_{x-2};$$

$$\therefore u_{x+3} - u_{x+2} - 3u_{x+1} - u_x = 0;$$

$$\therefore\ m^3 - m^2 - 3m - 1 = m(m^2 - 1) - (m+1)^2 = 0\,;$$

$$\therefore\ m = -1, \text{ and } 1 \pm \sqrt{2}\,;$$

$$\therefore\ u_x = a(-1)^x + b(1+\sqrt{2})^x + c(1-\sqrt{2})^x,$$

$$u_0 = 0 = a + b + c,$$

$$u_1 = 1 = -a + b(1+\sqrt{2}) + c(1-\sqrt{2}),$$

$$u_2 = 4 = a + b(3 + 2\sqrt{2}) + c(3 - 2\sqrt{2})\,;$$

$$\therefore\ a = 1,\quad b = \frac{1}{2\sqrt{2}}(3-\sqrt{2}),\quad c = -\frac{1}{2\sqrt{2}}(3+\sqrt{2})\,;$$

$$\therefore\ u_x = (-1)^x + \frac{1}{2\sqrt{2}}(3-\sqrt{2})(1+\sqrt{2})^x - \frac{1}{2\sqrt{2}}(3+\sqrt{2})(1-\sqrt{2})^x.$$

If the law of increase be one in the second and every succeeding year,

$$\sqrt{5}u_x = \frac{1}{2^x}(1+\sqrt{5})^x - \frac{1}{2^x}(1-\sqrt{5})^x.$$

3. If a person possessed of a given capital of £n, spends in the first year the whole of the interest receivable at the end of that year; and in the second year twice the interest receivable at the end of the second year; and in the third year thrice the interest receivable at the end of that year, and so on; to find how long it will be before his capital is exhausted.

Let u_x = reduced capital in pounds at the end of the x^{th} year, r = interest of £1 for a year; then ru_x = interest receivable at end of $(x+1)^{\text{th}}$ year, and $(x+1)\,ru_x$ = expenditure in $(x+1)^{\text{th}}$ year, therefore reduced capital at end of $(x+1)^{\text{th}}$ year, is

$$u_x + ru_x - (x+1)\,ru_x = (1 - rx)\,u_x\,;$$

$$\therefore\ u_{x+1} - (1 - rx)\,u_x = 0\,;$$

$$\therefore\ u_x = n(1-r)(1-2r)\ldots\{1-(x-1)\,r\},$$

since when $x = 1$, $u_x = n$. Now this vanishes when $x = 1 + \dfrac{1}{r}$, which expresses the number of years the capital will last.

100. The following are instances of Geometrical Problems, which admit of being solved by Finite Differences.

1. To find a curve in which the chord QP joining the extremities of any two ordinates PN, QM at a given distance from one another, when produced shall meet the axis of the abscissæ at a constant distance from the foot of either ordinate.

$$\text{Let } MN = h, \quad TN = \frac{h}{n-1}, \quad PN = y_x, \quad QM = y_{x+h}, \text{ (fig. 6)}$$

$$\text{then } (y_{x+h} - y_x) \div h = y_x \div \frac{h}{n-1};$$

$$\therefore\ y_{x+h} - ny_x = 0; \quad \therefore\ y_x = Cn^{\frac{x}{h}};$$

$$\text{or } y = Cn^{\frac{x}{h}} \text{ is the equation to the curve.}$$

2. To find a curve such that the portion of a straight line drawn through a fixed point in its plane and terminated by the curve, shall be of a constant length. Let $\rho = f(\theta)$ be its equation, then we must have for all values of θ,

$$f(\theta) + f(\theta + \pi) = c.$$

As π is here the increment of θ, let $\theta = \pi z$, and let $f(\pi z)$ be denoted by u_z, then $u_{z+1} + u_z = c$;

$$\therefore\ u_z = \tfrac{1}{2}c + C_z(-1)^z,$$

where C_z denotes any function of z that does not alter in value by the change of z into $z+1$. Now $(-1)^z = \cos \pi z$, and $F(\cos 2\pi z)$ fulfils the condition to which C_z is subject;

$$\therefore\ \rho = f(\theta) = \tfrac{1}{2}c + F(\cos 2\theta) \,.\, \cos\theta,$$

the equation to the required curve. If $F(\cos 2\theta) = b$, this becomes $\rho = \frac{1}{2}c + b\cos\theta$; which represents a circle if $b = 0$.

3. To find a curve such that any n radii drawn through a fixed point in its plane so as to make equal angles with one

another, and terminating in the curve, shall have their sum invariable.

If $\rho = f(\theta)$ be the equation to the curve, and h denote the angle between two consecutive radii, we have

$$f(\theta) + f(\theta + h) + \ldots + f\{\theta + (n-1)h\} = c \ldots (1),$$

or making $\theta = hz$, and $f(hz) = u_z$,

$$u_{z+n-1} + u_{z+n-2} + \ldots + u_z = c \ldots\ldots\ldots\ldots (1),$$

the solution of which is

$$u_z = \frac{1}{n}c + C_z' \cos\frac{2\pi z}{n} + C_z'' \cos\frac{4\pi z}{n}$$

$$+ C_z''' \cos\frac{6\pi z}{n} + \&c. + C_z^{(n-1)} \cos\frac{(2n-2)\pi z}{n},$$

for it is seen by summing the resulting series that any one of the values $\cos\frac{2\pi z}{n}$, $\cos\frac{4\pi z}{n}$, &c. for u_z, satisfies (1), when $c = 0$. Now suppose $h = \frac{2\pi}{m}$, then $\frac{2\pi z}{n} = \frac{mhz}{n} = \frac{m\theta}{n}$, and C_z' may be replaced by $f_1(\cos 2\pi z) = f_1(\cos m\theta)$, &c.; consequently the required equation to the curve is

$$\rho = \frac{1}{n}c + f_1(\cos m\theta)\cos\frac{m\theta}{n} + f_2(\cos m\theta)\cos\frac{2m\theta}{n} + \ldots$$

$$+ f_{n-1}(\cos m\theta)\cos\frac{(n-1)m\theta}{n}.$$

If $m = n$, $f_1(\cos m\theta) = b$, and if the rest of the arbitrary functions vanish, we get

$$\rho = \frac{1}{n}c + b\cos\theta.$$

4. To find a curve such that the product of the two segments of any straight line drawn through a fixed point in its plane to meet the curve, shall be constant. If $\rho = f(\theta)$ be the equation to the curve, then

$$f(\theta) \cdot f(\theta+\pi) = c^2; \quad \text{let } \theta = \pi z, \text{ and } f(\pi z) = u_z;$$

$$\therefore \; u_{z+1} \cdot u_z = c^2, \text{ which gives (Art. 86)}$$

$$u_z = c\,C_z^{(-1)^z} = c \cdot \{F(\cos 2\pi z)\}^{\cos \pi z},$$

since C_z is any function of z that does not alter in value when z is changed into $z+1$, and $(-1)^z = \cos \pi z$;

$$\therefore \; \rho = c\,\{F(\cos 2\theta)\}^{\cos\theta}.$$

Now $F(\cos 2\theta)$ denotes any function of $\cos 2\theta$ whose value does not alter by the change of θ into $\pi+\theta$; and may therefore be replaced by

$$\left\{\frac{F(\cos 2\theta)+\cos\theta}{F(\cos 2\theta)-\cos\theta}\right\}^{\frac{n}{\cos\theta}},$$

which evidently has the property of not being altered by the substitution of $\pi+\theta$ for θ;

$$\therefore \; \frac{\rho}{c} = \left\{\frac{F(\cos 2\theta)+\cos\theta}{F(\cos 2\theta)-\cos\theta}\right\}^{n},$$

which gives algebraic curves by assigning an algebraic form to the arbitrary function F. Thus if $n=\frac{1}{2}$, and

$$F(\cos 2\theta) = \frac{\sqrt{a^2-b^2\sin^2\theta}}{b}, \quad \text{then } \frac{\rho}{c} = \frac{\sqrt{a^2-b^2\sin^2\theta}+b\cos\theta}{\sqrt{a^2-b^2}},$$

the equation to a circle. Several other interesting Problems of this description may be found in Herschel's Examples.

The following example supplies an omission at the end of Art. 84*. It is evident that $v_x = a^x$ satisfies the equation

$$u_{x+2} - a\,(1+B_x)\,u_{x+1} + a^2 B_x u_x = 0,$$

$$\therefore \; w_x = CB_x \text{ and } u_x = a^x\,\{C' + C\Sigma\,(PB_{x-1})\}.$$

Suppose $B_x = \dfrac{x+1}{x+3}$, then $PB_{x-1} = \dfrac{2.3\ldots x}{4.5\ldots(x+2)} = \dfrac{1}{(x+1)(x+2)}$;

$\therefore \; \Sigma\,(PB_{x-1}) = \dfrac{C}{x+1}$; and $u_x = a^x\left(C' + \dfrac{C}{x+1}\right)$ is the solution of $(x+3)\,u_{x+2} - 2a\,(x+2)\,u_{x+1} + a^2\,(x+1)\,u_x = 0.$

SECTION IV.

SUMMATION OF SERIES.

Integration of the General Term.

101. ONE of the most direct and important applications of the Calculus of Finite Differences, is the general method which it furnishes of assigning the sum of any number of Terms of a Series, of the general term of which we are able to take the integral. There will be two cases to consider, according as the general term is given explicitly in terms of the index, or is only given by means of an equation of differences. We shall begin with the former case, by shewing that the sum of any number of terms ending with the general term u_x, is equal to the integral of the following term, together with a constant.

Let S_x denote the sum of the first x terms of a series whose general term is u_x,

$$\text{then } S_x = u_1 + u_2 + \ldots + u_x,$$

$$S_{x+1} = u_1 + u_2 + \ldots + u_x + u_{x+1};$$

$$\therefore \quad S_{x+1} - S_x = \Delta S_x = u_{x+1};$$

$$\therefore \; S_x = \Sigma u_{x+1} + C.$$

$$\text{Making } x = 0, \quad S_0 = 0 = \Sigma u_{x=1} + C;$$

or, as it is usually written, $0 = \Sigma u_1 + C$,

$$\therefore \; S_x = \Sigma u_{x+1} - \Sigma u_1.$$

In general the arbitrary constant will be determined by the term with which we make the series commence. Thus, if we

make it begin with u_r instead of u_1, this amounts to supposing the sum of the terms preceding u_r, that is, Σu_r to be zero; and therefore to determine the constant, we have

$$S_{r-1} = \Sigma u_{x=r} + C = 0.$$

102. As the summation of series, where the general term is given explicitly as a function of the index, thus resolves itself into the cases of integration treated of in Section II., it will only be necessary to give a few numerical examples; every expression integrated in that Section gives the sum of a series of which it is the general term.

1. To find the sum of the first x terms of any progression of figurate numbers. In the r^{th} order the general term is

$$u_x = \frac{x(x+1)(x+2)\dots(x+r-2)}{1.2.3\dots(r-1)};$$

$$\therefore\ \Sigma u_{x+1} = \frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r} + C, \quad \text{(Art. 45)}$$

$$\Sigma u_1 = 0 + C = 0;$$

$$\therefore\ S_x = \frac{x(x+1)\dots(x+r-1)}{1.2.3\dots r}.$$

Similarly, for the series of inverse figurate numbers, except when $r=2$, that is, for the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{x}$.

For we have $u_x = \dfrac{1.2.3\dots(r-1)}{x(x+1)\dots(x+r-2)}$;

$$\therefore\ \Sigma u_{x+1} = C - \frac{1.2.3\dots(r-1)}{(r-2)(x+1)\dots(x+r-2)}, \quad \text{(Art. 47)}$$

$$\Sigma u_1 = C - \frac{r-1}{r-2} = 0,$$

$$S_x = (r-1)\left\{\frac{1}{r-2} - \frac{1.2.3\ldots(r-3)}{(x+1)(x+2)\ldots(x+r-2)}\right\},$$

and $S_\infty = \dfrac{r-1}{r-2}$.

2. To find the sums of the squares and cubes of the natural numbers. Here $u_x = x^2 = (x-1)x + x$;

$$\therefore\ \Sigma u_{x+1} = \frac{(x-1)x(x+1)}{3} + \frac{x(x+1)}{2} + C;$$

and $\Sigma u_1 = 0 + C = 0$; $\therefore\ S_x = \dfrac{x(x+1)(2x+1)}{6}$.

Again, $u_x = x^3 = x(x^2-1) + x$;

$$\therefore\ \Sigma u_{x+1} = \frac{(x-1)x(x+1)(x+2)}{4} + \frac{x(x+1)}{2} + C,$$

$$\Sigma u_1 = 0 + C = 0;$$

$$\therefore\ S_x = \frac{x(x+1)}{2}\left\{\frac{(x-1)(x+2)}{2} + 1\right\} = \left\{\frac{x(x+1)}{2}\right\}^2.$$

Similarly, $1^2 + 3^2 + 5^2 + \ldots + (2x-1)^2 = \dfrac{x(4x^2-1)}{3}$.

$$1^3 + 3^3 + 5^3 + \ldots + (2x-1)^3 = 2x^4 - x^2.$$

3. To sum the series $1^2 - 2^2 + 3^2 - 4^2 + \ldots \pm x^2$.

Here $u_x = (-1)^{x-1}x^2$;

$$\Sigma u_x = \frac{(-1)^{x-1}x^2}{-2} - \frac{(2x+1)(-1)^x}{4} + \frac{2(-1)^{x+1}}{-8} + C;\ \text{(Art. 59)}$$

$$\therefore\ \Sigma u_{x+1} = (-1)^{x+1}\left\{\frac{(x+1)^2}{2} - \frac{2x+3}{4} + \frac{1}{4}\right\} + C;$$

$$\therefore\ S_x = (-1)^{x+1}\frac{x(x+1)}{2}.\quad \text{Similarly,}$$

$$2.1^2 + 2^2.2^2 + 2^3.3^2 + \ldots + 2^x x^2 = 2^{x+1}(x+1)(x-3) + (2^{x+1}-1)\,6.$$

4. To find the sum of x terms of the series

$$\frac{1}{2^2-3^2} + \frac{1}{4^2-3^2} + \frac{1}{6^2-3^2} + \&c.$$

$$u_x = \frac{1}{(2x)^2-3^2} = \frac{1}{(2x-3)(2x+3)},\ \text{which falls under Art. 48,}$$

$$S_x = \frac{1}{18} - \frac{x+1}{(2x+1)(2x+3)} - \frac{1}{6}.\frac{6x+5}{(2x-1)(2x+1)(2x+3)};$$

and $S_\infty = \frac{1}{18}$. Similarly,

$$\frac{5}{1.2.3} + \frac{8}{2.3.4} + \frac{11}{3.4.5} + \&c.\ \text{to } x \text{ terms} = 2 - \frac{3x+4}{(x+1)(x+2)}.$$

5. $2.4.7 + 4.7.13 + 8.13.25 + 16.25.49 + \&c.$ to x terms.

Here $u_x = 2^x(3.2^{x-1}+1)(3.2^x+1)$.

Assume $\Sigma u_{x+1} = A(3.2^{x-1}+1)(3.2^x+1)(3.2^{x+1}+1)$; (Art. 55)

$$\therefore\ u_{x+1} = A(3.2^x+1)(3.2^{x+1}+1)\{3.2^{x+2}+1-(3.2^{x-1}+1)\}$$

$$= \frac{21}{4}A.2^{x+1}(3.2^x+1)(3.2^{x+1}+1);\ \text{which gives } A = \frac{4}{21},$$

$$\therefore\ \Sigma u_{x+1} = \frac{4}{21}(3.2^{x-1}+1)(3.2^x+1)(3.2^{x+1}+1) + C;$$

$$\Sigma u_1 = \frac{4}{21}.\frac{5}{2}.4.7 + C = 0;$$

$$\therefore\ S_x = \frac{4}{21}\,(3\,.\,2^{x-1}+1)\,(3\,.\,2^x+1)\,(3\,.\,2^{x+1}+1) - \frac{40}{3}.$$

6. $\dfrac{2}{3\,.\,3} - \dfrac{4}{3\,.\,9} + \dfrac{8}{9\,.\,15} - \dfrac{16}{15\,.\,33} + \&c.$

$$\text{Here } u_x = \frac{(-2)^x}{\{(-2)^x-1\}\,\{(-2)^{x+1}-1\}};$$

$$\therefore\ S_x = \frac{1}{9} + \frac{1}{3}\cdot\frac{1}{(-2)^{x+1}-1},\ \text{(Art. 55), and } S_\infty = \frac{1}{9}.$$

7. $\dfrac{16}{2\,.\,3\,.\,4} - \dfrac{21}{3\,.\,4\,.\,5}\cdot\dfrac{2}{3} + \dfrac{26}{4\,.\,5\,.\,6}\cdot\left(\dfrac{2}{3}\right)^2 - \&c.$ to x terms,

$$\text{Here } u_x = \frac{16+5\,(x-1)}{(x+1)\,(x+2)\,(x+3)}\,(-\tfrac{2}{3})^{x-1}.$$

$$\text{Assume } \Sigma u_{x+1} = \frac{A}{(x+2)\,(x+3)}\,(-\tfrac{2}{3})^x, \quad \text{(Art. 56)}$$

$$\text{then } A = -3, \text{ and } \therefore\ S_x = \tfrac{1}{2} - \frac{3}{(x+2)\,(x+1)}\,(-\tfrac{2}{3})^x.$$

Similarly for the sum of x terms of the series

$$\frac{19}{1.2.3}\cdot\frac{1}{4} + \frac{28}{2.3.4}\cdot\frac{1}{8} + \frac{39}{3.4.5}\cdot\frac{1}{16} + \frac{52}{4.5.6}\cdot\frac{1}{32} + \dots$$

$$u_x = \frac{x^2+6x+12}{x\,(x+1)\,(x+2)}\,(\tfrac{1}{2})^{x+1},\quad S_x = 1 - \frac{x+4}{(x+1)\,(x+2)}\,(\tfrac{1}{2})^{x+1}.$$

8. $\dfrac{1}{\cos\theta\cos 2\theta} + \dfrac{1}{\cos 2\theta\cos 3\theta} + \dfrac{1}{\cos 3\theta\cos 4\theta} + \&c.$

to x terms.

$$\text{Here } u_x = \frac{1}{\cos x\theta\,\cos\,(x+1)\,\theta};\ \text{therefore (Art. 53)}$$

$$\Sigma u_{x+1} = \frac{\tan\,(x+1)\,\theta}{\sin\theta} + C,\quad \Sigma u_1 = \frac{\tan\theta}{\sin\theta} + C = 0,$$

$$\therefore\ S_x = \frac{\tan(x+1)\theta - \tan\theta}{\sin\theta}.$$

9. $1\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \&c.$, to x terms.

Here $u_{x+1} = (x+1)\cos(x+1)\theta$,

$$\Sigma u_{x+1} = (x+1)\frac{\sin(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} - \frac{1}{2\sin\frac{1}{2}\theta}\Sigma\sin(x+\tfrac{3}{2})\theta \text{ (Art. 59)}$$

$$= \frac{(x+1)\sin(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} + \frac{\cos(x+1)\theta}{(2\sin\frac{1}{2}\theta)^2} + C,$$

$$\Sigma u_1 = \tfrac{1}{2} + \frac{\cos\theta}{(2\sin\frac{1}{2}\theta)^2} + C = 0, \quad \therefore\ C = -\frac{1}{(2\sin\frac{1}{2}\theta)^2},$$

$$\therefore\ S_x = \frac{x\sin(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} + \frac{1}{2\sin^2\frac{1}{2}\theta}\{\sin(x+\tfrac{1}{2})\theta\sin\tfrac{1}{2}\theta - \sin^2\tfrac{1}{2}(x+1)\theta\}$$

$$= \frac{x\sin(x+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} - \frac{1}{2}\left(\frac{\sin\frac{1}{2}x\theta}{\sin\frac{1}{2}\theta}\right)^2.$$

Hence also, if $S_x = 1\cos^2\theta + 2\cos^2 2\theta + 3\cos^2 3\theta + \&c.$

then $2S_x = 1(1+\cos 2\theta) + 2(1+\cos 4\theta) + \&c.$

$$\therefore\ S_x = \frac{x(x+1)}{4} + \frac{x\sin(2x+1)\theta}{4\sin\theta} - \frac{1}{4}\left(\frac{\sin x\theta}{\sin\theta}\right)^2.$$

10. $a\cos\theta + 2a^2\cos 2\theta + 3a^3\cos 3\theta + \&c.$, to x terms.

Here $u_x = xa^x\cos x\theta = xv_x$, suppose;

$$\therefore\ \Sigma u_x = x\Sigma v_x - \Sigma^2 v_{x+1}$$

$$= \frac{x}{c}(a^2D^{-1} - 1)v_x - \frac{1}{c^2}(a^2D^{-1} - 1)^2 v_{x+1};\ \text{(Art. 52)}$$

$$\therefore\ \Sigma u_{x+1} = \frac{x+1}{c}(a^2v_x - v_{x+1}) - \frac{1}{c^2}(a^4v_x - 2a^2v_{x+1} + v_{x+2}) + C,$$

$$0 = \frac{1}{c}(a^2 - a\cos\theta) - \frac{1}{c^2}(a^4 - 2a^3\cos\theta + a^2\cos 2\theta) + C;$$

$$\therefore\ C = \frac{a}{c}\cos\theta - 2\left(\frac{a\sin\theta}{c}\right)^2;$$

$$\therefore\ S_x = \frac{x+1}{c}(a^2v_x - v_{x+1}) - \frac{1}{c^2}(a^4v_x - 2a^2v_{x+1} + v_{x+2}) + \frac{a}{c}\cos\theta - 2\frac{a^2}{c^2}\sin^2\theta;$$

where $v_x = a^x\cos x\theta$. Similarly, if the general term be $x^2a^x\cos x\theta$.

11. $\tan^{-1}\dfrac{1}{1+1+1^2} + \tan^{-1}\dfrac{1}{1+2+2^2} + \tan^{-1}\dfrac{1}{1+3+3^2} + \text{\&c.}$
to x terms;

$$u_x = \tan^{-1}\frac{1}{1+x+x^2} = \tan^{-1}\frac{\Delta x}{1+x(x+1)},$$

$$\therefore\ \Sigma u_x = \tan^{-1}x + C, \quad \text{(Art. 54)}$$

$$\therefore\ S_x = \Sigma u_{x+1} - \Sigma u_1 = \tan^{-1}(x+1) - \tfrac{1}{4}\pi.$$

12. $\tan^{-1}\left(\dfrac{\theta^3}{4+3\theta^2}\right) + 2\tan^{-1}\left(\dfrac{\theta^3}{32+6\theta^2}\right) + 4\tan^{-1}\left(\dfrac{\theta^3}{256+12\theta^2}\right) + \text{\&c.}$

Here $u_{x+1} = 2^x\tan^{-1}\dfrac{\theta^3}{4\,.\,2^{3x} + 3\,.\,2^x\theta^2};$

$$\therefore\ \Sigma u_{x+1} = 2^x\tan^{-1}\frac{\theta}{2^x} + C, \quad \text{(Art. 54)}$$

$$\therefore\ S_x = \Sigma u_{x+1} - \Sigma u_1 = 2^x\tan^{-1}\frac{\theta}{2^x} - \tan^{-1}\theta, \text{ and } S_\infty = \theta - \tan^{-1}\theta.$$

And we may similarly shew for the series whose $(x+1)^{\text{th}}$ term is

$$3^x\tan^{-1}\left(\frac{8\theta^3 3^x}{27\,.\,3^{4x} + 18\theta^2\,.\,3^{2x} - \theta^4}\right),$$

that $S_x = 3^x\tan^{-1}\dfrac{\theta}{3^x} - \tan^{-1}\theta$, and $S_\infty = \theta - \tan^{-1}\theta$.

13. $\log\cot\theta + \frac{1}{2}\log\cot 2\theta + \frac{1}{4}\cot 4\theta + \text{\&c.}$ to x terms.

Here $u_{x+1} = \dfrac{1}{2^x}\log\cot 2^x\theta = 2\Delta\dfrac{1}{2^x}\log(2\sin 2^x\theta);$

$$\therefore\ S_x = \frac{1}{2^{x-1}}\log(2\sin 2^x\theta) - 2\log(2\sin\theta).$$

Recurring Series.

103. We next come to the case where the general term is not given explicitly in terms of its index, but only certain relations between the consecutive terms, or these and their indices are expressed.

The equation of differences which determines the form of the general term, cannot always be solved; when however the equation is linear with constant coefficients, its solution, as we know, can be effected; and this happens for Recurring Series—the most remarkable class of this sort of series.

This equation when integrated will involve the same number of arbitrary constants as is expressed by its order: implying that that number of consecutive terms may be regarded as indeterminate, and the remaining terms formed according to the law which the equation to the series expresses.

A recurring series is a series in which an equation of the first degree with constant coefficients, holds good between a certain definite number of consecutive terms, in whatever part of the series they be taken. For example, in the series

$$3 + 5 + 9 + 17 + 33 + \&c.$$

we have $9 = 3 \,.\, 5 - 2 \,.\, 3$, $17 = 3 \,.\, 9 - 2 \,.\, 5$, &c.; and in general

$$u_{x+2} = 3u_{x+1} - 2u_x. \quad (1)$$

The integral of this is $u_x = c \,.\, 2^x + c'$; and assuming a and b for the first two terms of the series of which u_x is the general term,

$$a = 2c + c', \; b = 4c + c'; \quad \therefore \; c = \tfrac{1}{2}(b - a), \; c' = 2a - b,$$

$$\text{so that } u_x = (b - a)\, 2^{x-1} + 2a - b;$$

and the series that has (1) for its equation is, in its most general form, $a + b + (3b - 2a) + (7b - 6a) + \&c.$

But for the series $\dfrac{a}{n+1} + \dfrac{a^2}{n+2} + \dfrac{a^3}{n+3} + \&c.$ we have

$$(x + n + 2)\, u_{x+2} - 2a\,(x + n + 1)\, u_{x+1} + a^2\,(x + n)\, u_x = 0,$$

an equation with variable coefficients, giving $u_x = \dfrac{a^x\,(c + c'x)}{x + n}$.

104. The general equation of every recurring series is

$$u_{x+n} + p_1 u_{x+n-1} + p_2 u_{x+n-2} + \dots + p_{n-1} u_{x+1} + p_n u_x = 0.$$

The series of coefficients which connects any term with the preceding ones is called the Scale of Relation. Thus

$$1 + p_1 + p_2 + \dots + p_n = 0$$

is the scale of relation of the recurring series whose equation is

$$u_{x+n} + p_1 u_{x+n-1} + \dots + p_n u_x = 0 \dots\dots\dots\dots (1).$$

105. A recurring series may generally be resolved into two or more geometric progressions.

For if $a_1, a_2, a_3, \dots a_n$ be the roots of the equation

$$a^n + p_1 a^{n-1} + p_2 a^{n-2} + \dots + p_n = 0 \dots\dots\dots\dots (2),$$

the complete integral of the equation of the series is

$$u_x = c_1 a_1^x + c_2 a_2^x + \dots + c_n a_n^x, \text{ (Art. 77)}$$

$$\therefore\ u_1 = c_1 a_1 + c_2 a_2 + c_3 a_3 + \dots + c_n a_n,$$

$$u_2 = c_1 a_1^2 + c_2 a_2^2 + c_3 a_3^2 + \dots + c_n a_n^2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_n = c_1 a_1^n + c_2 a_2^n + c_3 a_3^n + \dots c_n a_n^n,$$

and the proposed series consequently is transformed into

$$c_1 (a_1 + a_1^2 + \dots + a_1^n) + c_2 (a_2 + a_2^2 + \dots + a_2^n) + \&c.$$
$$+ c_n (a_n + a_n^2 + \dots a_n^n).$$

106. In the particular cases in which equation (2) has equal or impossible roots, the recurring series can no longer be resolved into geometric progressions; for the complete integral of the equation of the series becomes in those two cases, respectively,

$$u_x = (c_0 + c_1 x + c_2 x^2 + \dots + c_{r-1} x^{r-1})\, a_1^x + c_{r+1} a^x{}_{r+1} + \dots + c_n a_n^x,$$

$$u_x = (c_0 + c_1 x + \dots + c_{r-1} x^{r-1})\, \rho^x \cos x\theta$$
$$+ (c'_0 + c'_1 x + \dots + c'_{r-1} x^{r-1})\, \rho^x \sin x\theta + c_{2r+1} a^x{}_{2r+1} + \dots + c_n a_n^x.$$

107. Hence, to find the general term of a recurring series, we must integrate the equation expressing the relation between its successive terms, and determine the arbitrary constants by making the general term u_x coincide with a sufficient number of given terms. When a series is known to be recurring, its equation may be determined by assuming it to be of the form (1); and then forming a sufficient number of equations for finding the coefficients p_1, p_2, &c. p_n, by substituting the given terms in order.

108. To find the sum of x terms of a recurring series.

First, let its general term be of the form

$$u_x = c_1 a_1^x + c_2 a_2^x + \dots + c_n a_n^x \text{; then } S_x = \Sigma u_{x+1} - \Sigma u_1$$

$$= c_1 \cdot \frac{a_1^{x+1} - a_1}{a_1 - 1} + c_2 \frac{a_2^{x+1} - a_2}{a_2 - 1} + \dots + c_n \cdot \frac{a_n^{x+1} - a_n}{a_n - 1}.$$

Secondly, let the general term be

$$u_x = (c_0 + c_1 x + c_2 x^2 + \dots + c_{r-1} x^{r-1})\, a_1^x + c_{r+1} a^x_{r+1} + \&\text{c.};$$

then since

$$\Sigma x^n a^x = \frac{a^x x^n}{a-1} - \frac{a^{x+1} \Delta x^n}{(a-1)^2} + \frac{a^{x+2} \Delta^2 x^n}{(a-1)^3} - \&\text{c. (Art. 59)},$$

$$\Sigma u_{x+1} = c_0 \frac{a_1^{x+1}}{a_1 - 1} + c_1 \left\{ \frac{a_1^{x+1}(x+1)}{a_1 - 1} - \frac{a_1^{x+2}}{(a_1 - 1)^2} \right\} + \&\text{c.};$$

$$\therefore\ S_x = \Sigma u_{x+1} - \Sigma u_1, \text{ is determined.}$$

Thirdly, let

$$u_x = \rho^x (c_0 \cos x\theta + c_0' \sin x\theta) + \rho^x x (c_1 \cos x\theta + c_1' \sin x\theta)$$
$$+ \rho^x x^2 (c_2 \cos x\theta + c_2' \sin x\theta) + \&\text{c.},$$

then each term to be integrated will be of the form

$$x^n \rho^x \cos(x\theta + \alpha),$$

the integral of which may be found by Art. 58, because

$$\Sigma^n \rho^x \cos (x\theta + \alpha)$$

is always assignable.

Ex. 1. To find the sum of x terms of the series

$$1 + 5 + 17 + 53 + \&c.$$

Let the equation be $u_{x+2} + pu_{x+1} + qu_x = 0$;

then $17 + 5p + q = 0$, $53 + 17p + 5q = 0$, which give $p = -4$,

$$q = 3, \text{ so that } u_{x+2} - 4u_{x+1} + 3u_x = 0.$$

Let $u_x = a^x$; then $a^2 - 4a + 3 = 0$; $a = 3$ or 1;

$$\therefore\ u_x = c_1 3^x + c_2,\quad 1 = 3c_1 + c_2,\quad 5 = 9c_1 + c_2,$$

which give $c_1 = \frac{2}{3}$, $c_2 = -1$, so that $u_x = 2\,.\,3^{x-1} - 1$;

$$\therefore\ S_x = 2\,.\,\frac{3^x}{3-1} - x + C,\quad 0 = 1 + C;$$

$$\therefore\ S_x = 3^x - x - 1.$$

Ex. 2. $2 - a - a^2 + 2a^3 - a^4 - a^5 + 2a^6 - a^7 - \&c.$

$$u_x = 2a^{x-1} \cos \frac{(2x-2)\,\pi}{3};$$

$$S_x = \frac{2a^{x+1} \cos \frac{1}{3}(2x-2)\,\pi - 2a^x \cos \frac{2}{3}\pi x + a + 2}{a^2 + a + 1}.$$

Ex. 3. $1 + 2 + 3 + 8 + 13 + 30 + 55 + 116 + \&c.$

$$u_{x+3} - 3u_{x+1} - 2u_x = 0,\quad u_x = \frac{4}{9} 2^x + \frac{1}{9}(3x-4)(-1)^x.$$

$$S_x = \frac{8}{9} 2^x - \frac{3}{4} + (-1)^x . \frac{6x-5}{36}.$$

Ex. 4. $1 + 4 + 18 + 80 + 356 + \&c.$, to x terms.

$$\text{Here } u_{x+2} - 4u_{x+1} - 2u_x = 0\,;$$

$$\therefore\ 2\sqrt{6}u_x = (2+\sqrt{6})^x - (2-\sqrt{6})^x;\ \text{and}$$

$$2\sqrt{6}S_x = \frac{(2+\sqrt{6})^{x+1} - (2+\sqrt{6})}{1+\sqrt{6}} - \frac{(2-\sqrt{6})^{x+1} - (2-\sqrt{6})}{1-\sqrt{6}}.$$

Application of the Integral Calculus to the Summation of Series.

109. As integrals are often expressed by Series, so, conversely, the latter may be represented by integrals; and it is often desirable to find the integral of which a proposed series is one of the developments, in order to subject it to the methods which we possess for calculating, at least approximately, the value of any integral taken between assigned limits. We proceed therefore to notice one or two processes given by Euler for effecting this; they consist chiefly in performing certain operations on the series, by which it is transformed into another series which we are able to sum, or which is similar to the proposed one.

110. Series which proceed according to the powers of some quantity t, affected with coefficients consisting of factors in arithmetic progression either in the numerator or denominator, may be summed by the aid of the Integral Calculus, the denominators being taken away by differentiation and the numerators by integration.

Ex. 1. Let $s = \frac{t}{2} + \frac{2t^2}{3} + \frac{3t^3}{4} + \&\text{c. (to } \infty\text{)},$

$$\therefore\ \frac{d}{dt}(st) = t + 2t^2 + 3t^3 + \&\text{c.}$$

$$\therefore\ \int dt\,.\frac{1}{t}\frac{d}{dt}(st) = t + t^2 + t^3 + \&\text{c.} = \frac{t}{1-t};$$

$$\therefore\ \frac{1}{t}\frac{d\,(st)}{dt} = \frac{1}{(1-t)^2},\quad st = \int dt\,\frac{t}{(1-t)^2} = \log\,(1-t) + \frac{1}{1-t} + C,$$

$$\therefore\ s = \frac{\log\,(1-t)}{t} + \frac{1}{1-t},\ \text{since } C = -1.$$

Ex. 2. $s = \frac{a+b}{\alpha+\beta} t + \frac{a+2b}{\alpha+2\beta} t^2 + \dots \frac{a+nb}{\alpha+n\beta} t^n + \dots$ to m terms;

$$\therefore\ \beta \frac{d}{dt} (st^{\frac{\alpha}{\beta}}) = \dots + (a+nb)\, t^{n+\frac{\alpha}{\beta}-1} + \dots$$

$$\frac{\beta}{b} \int dt \left\{ t^{\frac{a}{b}-\frac{\alpha}{\beta}} \frac{d}{dt} (st^{\frac{\alpha}{\beta}}) \right\} = \dots + t^{n+\frac{a}{b}} + \dots = \frac{t^m - 1}{t-1}\, t^{\frac{a}{b}+1};$$

$$\therefore\ s = \frac{b}{\beta} t^{-\frac{\alpha}{\beta}} \int dt\, t^{\frac{\alpha}{\beta}-\frac{a}{b}} \cdot \frac{d}{dt} \left\{ \frac{t^m - 1}{t-1} \cdot t^{\frac{a}{b}+1} \right\}.$$

111. Series like the above in which the coefficient of every term is the same function of the index of that term, may be readily summed by separation of symbols. Thus if we put $t = e^{\theta}$, and denote $\frac{d}{d\theta}$ by d, so that $f(d)\, e^{m\theta} = f(m)\, e^{m\theta}$, the series in Ex. 2, taken to infinity, may be written

$$s = \frac{a+bd}{\alpha+\beta d} (e^{\theta} + e^{2\theta} + e^{3\theta} + \dots) = \frac{a+bd}{\alpha+\beta d} \frac{e^{\theta}}{1-e^{\theta}}$$

$$= \left\{ \frac{a\beta - b\alpha}{\beta} (\alpha + \beta d)^{-1} + \frac{b}{\beta} \right\} \frac{e^{\theta}}{1-e^{\theta}}$$

$$= \frac{a\beta - b\alpha}{\beta^2} e^{-\frac{\alpha\theta}{\beta}} \int d\theta \frac{e^{\left(\frac{\alpha}{\beta}+1\right)\theta}}{1-e^{\theta}} + \frac{b}{\beta} \cdot \frac{e^{\theta}}{1-e^{\theta}}$$

$$= \frac{a\beta - b\alpha}{\beta^2} t^{-\frac{\alpha}{\beta}} \int dt\ \frac{t^{\frac{\alpha}{\beta}}}{1-t} + \frac{b}{\beta} \cdot \frac{t}{1-t},$$

which may be easily shewn to coincide with the preceding result when m is infinite. And in general the series

$$f(n)\, t^n + f(n+r)\, t^{n+r} + f(n+2r)\, t^{n+2r} + \text{\&c. to } m \text{ terms},$$

becomes by putting $t = e^{\theta}$, $\frac{d}{d\theta} = d$,

$$f(d)\, (e^{n\theta} + e^{(n+r)\theta} + e^{(n+2r)\theta} + \text{\&c.}) = f(d)\, e^{n\theta} \cdot \frac{1 - e^{mr\theta}}{1 - e^{r\theta}}.$$

Thus the infinite series

$$\frac{5t^3}{1.2.3}+\frac{6t^4}{2.3.4}+\frac{7t^5}{3.4.5}+\text{\&c. becomes}$$

$$\frac{d+2}{d(d-1)(d-2)}(e^{3\theta}+e^{4\theta}+e^{5\theta}+\text{\&c.})=\frac{d+2}{d(d-1)(d-2)}\,\frac{e^{3\theta}}{1-e^{\theta}}$$

$$=\left(\frac{1}{d}-\frac{3}{d-1}+\frac{2}{d-2}\right)\frac{e^{3\theta}}{1-e^{\theta}}$$

$$=\int d\theta\,\frac{e^{3\theta}}{1-e^{\theta}}-3e^{\theta}\int d\theta\,\frac{e^{2\theta}}{1-e^{\theta}}+2e^{2\theta}\int d\theta\,\frac{e^{\theta}}{1-e^{\theta}}$$

$$=\int dt\,\frac{t^2}{1-t}-3t\int dt\,\frac{t}{1-t}+2t^2\int dt\,\frac{1}{1-t}$$

$$=\frac{5t^2}{2}-t-(1-3t+2t^2)\log(1-t).$$

112. In the following instances the coefficient of any term of the series is not a function of the index of an invariable form.

Ex. 1. $$s=1+\frac{a}{\alpha+\beta}t+\frac{a(a+b)}{(\alpha+\beta)(\alpha+2\beta)}t^2$$

$$+\frac{a(a+b)(a+2b)}{(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}t^3+\text{\&c. (to }\infty),$$

$$\frac{1}{b}\int dt\,st^{\frac{a}{b}-2}=\frac{t^{\frac{a}{b}-1}}{a-b}+\frac{t^{\frac{a}{b}}}{\alpha+\beta}+\frac{a}{(\alpha+\beta)(\alpha+2\beta)}t^{\frac{a}{b}+1}$$

$$+\frac{a(a+b)}{(\alpha+\beta)(\alpha+2\beta)(\alpha+3\beta)}t^{\frac{a}{b}+2}+\text{\&c.};$$

$$\therefore\ \frac{\beta}{b}\frac{d}{dt}\left(t^{\frac{\alpha}{\beta}-\frac{a}{b}+1}.\int dt\,st^{\frac{a}{b}-2}\right)=\frac{\alpha}{a-b}t^{\frac{\alpha}{\beta}-1}+t^{\frac{\alpha}{\beta}}+\frac{a}{\alpha+\beta}t^{\frac{\alpha}{\beta}+1}$$

$$+\frac{a(a+b)}{(\alpha+\beta)(\alpha+2\beta)}t^{\frac{\alpha}{\beta}+2}+\text{\&c.}=\frac{\alpha}{a-b}t^{\frac{\alpha}{\beta}-1}+t^{\frac{\alpha}{\beta}}s;\ldots(1)$$

hence, performing the differentiation, we find

$$(\beta t - bt^2)\frac{ds}{dt} + (\alpha - at)\,s = \alpha,$$

a linear equation of the first order for finding s.

For the sum of m terms, we shall evidently have from equation (1)

$$\frac{\beta}{b}\frac{d}{dt}\left(t^{\frac{\alpha}{\beta}-\frac{a}{b}+1}\int dt\, s t^{\frac{a}{b}-2}\right) = \frac{\alpha}{a-b}\, t^{\frac{\alpha}{\beta}-1}$$

$$+\, t^{\frac{\alpha}{\beta}}\left\{s - \frac{a\,(a+b)\,\ldots\,\{a+(m-2)\,b\}}{(\alpha+\beta)\,(\alpha+2\beta)\,\ldots\,\{\alpha+(m-1)\,\beta\}}\,t^{m-1}\right\}.$$

Ex. 2. $\dfrac{t^a}{a} + \dfrac{t^{a+b}}{a\,(a+b)} + \dfrac{t^{a+2b}}{a\,(a+b)\,(a+2b)} + \text{\&c. (to } \infty\text{)}.$

$$s = e^{\frac{1}{b}t}\left(c + \int dt e^{-\frac{1}{b}t^b}\,.\,t^{a-1}\right).$$

If we suppose $a = b = 3$, we get $s = e^{\frac{1}{3}t^3} - 1.$

$$\text{Hence } \frac{1}{3} + \frac{1}{3\,.\,6} + \frac{1}{3\,.\,6\,.\,9} + \text{\&c.} = \sqrt[3]{e} - 1.$$

Ex. 3. $s = 1 + \dfrac{\alpha}{1}\,.\,\dfrac{\beta}{\gamma}\,t + \dfrac{\alpha\,(\alpha+1)}{1\,.\,2}\;\dfrac{\beta\,(\beta+1)}{\gamma\,(\gamma+1)}\,t^2$

$$+\,\frac{\alpha\,(\alpha+1)\,(\alpha+2)}{1\,.\,2\,.\,3}\;\frac{\beta\,(\beta+1)\,(\beta+2)}{\gamma\,(\gamma+1)\,(\gamma+2)}\,t^3 + \text{\&c. (to } \infty\text{)}.$$

$$(t^2 - t)\,\frac{d^2s}{dt^2} + \{(\alpha+\beta+1)\,t - \gamma\}\,\frac{ds}{dt} + \alpha\beta s = 0,$$

which is satisfied by $s = (1-t)^{-\alpha}$, when $\beta = \gamma$.

113. When consecutive denominators have only a single factor in common, or none at all, the following is a convenient mode of summing the series.

Ex. $\dfrac{1}{a\,(a+b)\,(a+2b)} + \dfrac{1}{(a+2b)\,(a+3b)\,(a+4b)}$

$$+\ \text{\&c. (to } \infty\text{)}.$$

Since $\dfrac{2b^2}{(a+xb)\{a+(x+1)b\}\{a+(x+2)b\}} = \dfrac{1}{a+xb}$

$$-\frac{2}{a+(x+1)b}+\frac{1}{a+(x+2)b},$$

resolving all the terms by means of this formula, by putting $x = 0, 2, 4$, &c., we find

$$2b^2S = \frac{1}{a} - \frac{2}{a+b} + \frac{2}{a+2b} - \frac{2}{a+3b} + \frac{2}{a+4b} - \text{\&c.}$$

$$\therefore\ 2b^2S = 2\int_0^1 dx \frac{x^{a-1}}{1+x^b} - \frac{1}{a}.$$

Similarly, $\dfrac{1}{a(a+b)(a+2b)(a+3b)}$

$$+\frac{1}{(a+3b)(a+4b)(a+5b)(a+6b)} + \text{\&c. (to } \infty)$$

$$= \frac{1}{6b^3}\left(\frac{1}{a} - 3\int_0^1 dx \frac{x^{a+b-1}}{1+x^b+x^{2b}}\right).$$

114. We may obtain an expression for the sum of the series

$$1 + \frac{x}{1}\cdot\frac{d\phi(z)}{dz} + \frac{x^2}{1.2}\frac{d^2\{\phi(z)\}^2}{dz^2} + \frac{x^3}{1.2.3}\frac{d^3\{\phi(z)\}^3}{dz^3} + \text{\&c.},$$

where $\phi(z)$ is any function of z, by Lagrange's Theorem.

For if $y = z + x\phi(y)$, then

$$y = z + \frac{x}{1}\phi(z) + \frac{x^2}{1.2}\frac{d\{\phi(z)\}^2}{dz} + \text{\&c.}$$

$$\therefore\ 1 + \frac{x}{1}\frac{d\phi(z)}{dz} + \frac{x^2}{1.2}\frac{d^2\{\phi(z)\}^2}{dz^2} + \text{\&c.} = \frac{dy}{dz} = \frac{1}{1 - x\dfrac{d}{dy}\phi(y)},$$

where $\dfrac{d}{dy}\phi(y)$ must be obtained in terms of x and z, from the equation $y = z + x\phi(y)$. Suppose, for example,

$$\phi(y) = \tfrac{1}{2}(y^2-1),\ \text{then}\ \frac{d\phi(y)}{dy} = y,$$

$$\text{and } y = z + \frac{x}{2}(y^2 - 1), \quad \therefore\ 1 - xy = \sqrt{1 - 2zx + x^2};$$

$$\therefore\ 1 + \frac{x}{1}\cdot\frac{1}{2}\frac{d(z^2-1)}{dz} + \frac{x^2}{1.2}\cdot\frac{1}{2^2}\cdot\frac{d^2(z^2-1)^2}{dz^2} + \ldots = \frac{1}{\sqrt{1-2zx+x^2}}.$$

Hence if

$$(1 - 2zx + x^2)^{-\frac{1}{2}} = 1 + Z_1 x + Z_2 x^2 + \ldots + Z_n x^n + \ldots,$$

$$\text{then } Z_n = \frac{1}{2^n \underline{|n}} \frac{d^n (z^2-1)^n}{dz^n}.$$

115. The function Z_n has some remarkable properties, of which the following may be noticed.

From the formula

$$\int dx \frac{1}{\sqrt{(a+bx)(c+ex)}} = \frac{2}{\sqrt{eb}} \log\left\{\sqrt{a+bx} + \sqrt{\frac{b}{e}}\,(c+ex)\right\}$$

it is easily shewn that

$$\int_{-1}^{+1} dz\,(1 - 2arz + a^2r^2)^{-\frac{1}{2}}\cdot\left(1 - \frac{2a}{r}z + \frac{a^2}{r^2}\right)^{-\frac{1}{2}} = \frac{1}{a}\log\left(\frac{1+a}{1-a}\right),$$

$$\therefore \int_{-1}^{+1} dz\,(1 + Z_1 ar + Z_2 a^2 r^2 + \ldots)\cdot\left(1 + Z_1\frac{a}{r} + Z_2\frac{a^2}{r^2} + \ldots\right)$$

$$= \frac{2}{a}\left(a + \frac{a^3}{3} + \frac{a^5}{5} + \&c.\right).$$

Hence the definite integral of every term of the product forming the first member that is not independent of r, must equal zero;

$$\therefore \int_{-1}^{+1} dz\,(Z_n Z_m) = 0, \text{ and } \int_{-1}^{+1} dz\,(Z_n)^2 = \frac{2}{2n+1}.$$

116. To find an expression for the sum of the reciprocals of the n^{th} powers of the values of y, in the equation

$$z - y + \phi(y) = 0;$$

where $\phi(y)$ denotes any function of y, and z a quantity independent of y.

Let $z - y + \phi(y) = C(a_1 - y)(a_2 - y) \dots (a_m - y)$,

$$\therefore \frac{1 - \frac{d}{dy}\phi(y)}{z - y + \phi(y)} = \frac{1}{a_1 - y} + \frac{1}{a_2 - y} + \dots + \frac{1}{a_m - y} \,.\, \dots\dots \text{(1)}$$

Now developing the two members of this equation in powers of y, the general term of the second member is

$$\left(\frac{1}{a_1^{n+1}} + \frac{1}{a_2^{n+1}} + \dots + \frac{1}{a_m^{n+1}}\right) y^n = s_{-n-1} \,.\, y^n,$$

and by Taylor's theorem the general term of the first member, putting $1 - \frac{d}{dy}\phi(y) = f(y)$, and regarding $f(y) \div (z - y)$ as a function of z and $\phi(y)$ as the increment of z, is

$$\frac{f(y)\,\phi^r(y)}{\lfloor r} \, \frac{d^r}{dz^r}\left(\frac{1}{z - y}\right).$$

But if $f(y) = A_0 + A_1 y + \dots + A_n y^n + \dots$,

$$\text{since } \frac{1}{z - y} = \frac{1}{z} + \frac{y}{z^2} + \dots + \frac{y^n}{z^{n+1}} + \dots,$$

$$\therefore \frac{f(y)}{z - y} = \dots + \frac{A_0 + A_1 z + \dots + A_n z^n}{z^{n+1}} y^n + \dots,$$

therefore the general term of $\frac{f(y)}{z - y}$ is $\frac{f(z)}{z^{n+1}} y^n$, whatever be the form of $f(z)$, if $\frac{f(z)}{z^{n+1}}$ be restricted to such terms only as involve negative powers of z; consequently the general term of $\frac{f(y)\,\phi^r(y)}{z - y}$ is $\frac{f(z)\,\phi^r(z)}{z^{n+1}} \,.\, y^n$, and therefore of

$$\frac{1}{\lfloor r} \frac{d^r}{dz^r} \left\{\frac{f(y)\,\phi^r(y)}{z - y}\right\} \text{ it is } \frac{1}{\lfloor r} \frac{d^r}{dz^r} \left\{\frac{f(z)\,\phi^r(z)}{z^{n+1}}\right\} y^n;$$

hence equating coefficients of y^n on both sides of equation (1),

$$s_{-n-1} = \frac{f(z)}{z^{n+1}} + \frac{d}{dz}\left\{\frac{f(z)\,\phi(z)}{z^{n+1}}\right\} + \frac{1}{1\,.\,2} \frac{d^2}{dz^2}\left\{\frac{f(z)\,\phi^2(z)}{z^{n+1}}\right\} + \&c.$$

Now putting for $f(z)$ its value, the general term may be resolved into

$$\frac{1}{\lfloor r}\frac{d^r}{dz^r}\left\{\frac{\phi^r(z)}{z^{n+1}}\right\}-\frac{1}{\lfloor r}\frac{d^r}{dz^r}\left\{\frac{\phi'(z)\,.\,\phi^r(z)}{z^{n+1}}\right\},$$

$$\text{or } \frac{1}{\lfloor r-1}\frac{d^{r-1}}{dz^{r-1}}\left\{\frac{\phi^{r-1}(z)\,\phi'(z)}{z^{n+1}}\right\}-\frac{n+1}{\lfloor r}\frac{d^{r-1}}{dz^{r-1}}\left\{\frac{\phi^r(z)}{z^{n+2}}\right\}$$

$$-\frac{1}{\lfloor r}\frac{d^r}{dz^r}\left\{\frac{\phi^r(z)\,\phi'(z)}{z^{n+1}}\right\},$$

of which the first and last will be destroyed by the corresponding parts in the preceding and succeeding terms; therefore, changing n into $n-1$,

$$s_{-n}=\frac{1}{z^n}-\frac{n}{1}\frac{\phi(z)}{z^{n+1}}-\frac{n}{1\,.\,2}\frac{d}{dz}\left\{\frac{\phi^2(z)}{z^{n+1}}\right\}-\ldots-\frac{n}{\lfloor r}\frac{d^{r-1}}{dz^{r-1}}\left\{\frac{\phi^r(z)}{z^{n+1}}\right\}-\&c.$$

But the second member of this equation is what Lagrange's Theorem gives for the development of y^{-n}, in the equation $z-y+\phi(y)=0$; therefore s_{-n} is equal to the sum of those terms of the development of y^{-n} which involve negative powers of z.

117. By the help of the preceding Proposition it may be shewn that Lagrange's Theorem, applied to the solution of the equation

$$z-y+\phi(y)=0,$$

gives an approximation to the least root. By what precedes, we have

$$\frac{s_{-n}}{s_{-n-r}}=\frac{\dfrac{1}{z^n}-\dfrac{n}{1}\dfrac{\phi(z)}{z^{n+1}}-\dfrac{n}{1\,.\,2}\dfrac{d}{dz}\left\{\dfrac{\phi^2(z)}{z^{n+1}}\right\}-\&c.}{\dfrac{1}{z^{n+r}}-\dfrac{n+r}{1}\dfrac{\phi(z)}{z^{n+r+1}}-\dfrac{n+r}{1\,.\,2}\dfrac{d}{dz}\left\{\dfrac{\phi^2(z)}{z^{n+r+1}}\right\}-\&c.},$$

where each series is restricted to those terms which involve negative powers of z; and the number of those terms increases with n, and if n be supposed very great, then each series may

be taken ad infinitum. But whatever be the value of n, if both series go on ad infinitum, the value of the second member is

$$\frac{y^{-n}}{y^{-n-r}} = y^r = z^r + \frac{r}{1} z^{r-1} \phi(z) + \frac{r}{1\,.\,2} \frac{d}{dz} \{z^{r-1} \phi^2(z)\} + \text{\&c.},$$

as given by Lagrange's Theorem, and which may be verified by actual multiplication; therefore when n is infinite,

$$\text{limit of } \frac{s_{-n}}{s_{-n-r}} = y^r,$$

(y^r denoting that function of z which is given for the development of y^r by Lagrange's Theorem).

But if a_1 be the least of the roots a_1, a_2, a_3, &c., we have also (Theory of Algebraical Equations, Art. 156)

$$\text{limit of } \frac{s_{-n}}{s_{-n-r}} = a_1^r, \text{ when } n \text{ is infinite};$$

$$\therefore\ y^r = a_1^r, \text{ and } y = a_1 \text{ the least root.}$$

118. To find the sums of the series

$$\frac{\sin\alpha}{1^2+k^2} + \frac{2\sin 2\alpha}{2^2+k^2} + \frac{3\sin 3\alpha}{3^2+k^2} + \text{\&c. (to } \infty\text{)},$$

$$\frac{\cos\alpha}{1^2+k^2} + \frac{\cos 2\alpha}{2^2+k^2} + \frac{\cos 3\alpha}{3^2+k^2} + \text{\&c. (to } \infty\text{)},$$

$$\text{or the values of } {}^1S^\infty \frac{n\sin n\alpha}{n^2+k^2}, \quad {}^1S^\infty \frac{\cos n\alpha}{n^2+k^2}.$$

First, we have (*Trigon.* Art. 155)

$$\frac{(1-p^2)\, e^{-kx}}{1-2p\cos x+p^2} = e^{-kx} + 2\,(p\cos x + p^2\cos 2x + p^3\cos 3x + \text{\&c.})\, e^{-kx},$$

and integrating both sides from $x=0$ to $x=\alpha$, and putting $p=1$,

we get $$\pi = \frac{1}{k}(1-e^{-k\alpha}) +$$

$$2e^{-k\alpha}\, {}^1S^\infty \frac{n\sin n\alpha}{k^2+n^2} - 2ke^{-k\alpha}\, {}^1S^\infty \frac{\cos n\alpha}{k^2+n^2} + 2\, {}^1S^\infty \frac{k}{k^2+n^2}.$$

Again, integrating from $x=0$ to $x=2\pi$, and putting $p=1$, we find

$$\pi\left(e^{-2k\pi}+1\right)=\frac{1-e^{-2k\pi}}{k}+2\,{}^1S^{\infty}\,\frac{k}{k^2+n^2}-2e^{-2k\pi}\,{}^1S^{\infty}\,\frac{k}{k^2+n^2}.$$

$$\text{which gives } 2\,{}^1S^{\infty}\,\frac{k}{k^2+n^2}=\pi\,.\,\frac{1+e^{-2k\pi}}{1-e^{-2k\pi}}-\frac{1}{k},$$

and substituting this value in the foregoing result, and calling the two sums we are in search of, S_1 and S_2, we have

$$\frac{2\pi e^{-2k\pi+k\alpha}}{1-e^{-2k\pi}}=\frac{1}{k}-2S_1+2kS_2;$$

and changing the sign of k,

$$\frac{-2\pi e^{-k\alpha}}{1-e^{-2k\pi}}=-\frac{1}{k}-2S_1-2kS_2;$$

therefore, adding and subtracting,

$$S_1={}^1S^{\infty}\,\frac{n\sin n\alpha}{k^2+n^2}=\frac{\pi}{2}\,\frac{e^{k(\pi-\alpha)}-e^{-k(\pi-\alpha)}}{e^{k\pi}-e^{-k\pi}},$$

$$S_2={}^1S^{\infty}\,\frac{\cos n\alpha}{k^2+n^2}=\frac{\pi}{2k}\,\frac{e^{k(\pi-\alpha)}+e^{-k(\pi-\alpha)}}{e^{k\pi}-e^{-k\pi}}-\frac{1}{2k^2}.$$

These formulæ were first given by Poisson, and may be considered as embracing the chief results which have hitherto been obtained relating to the summation of series of the sines and cosines of multiple angles.

119. The definite integrals required in the preceding investigation may be found as follows. Suppose

$$y=\frac{(1-p^2)e^{-kx}}{1-2p\cos x+p^2},$$

to be the equation to a curve; and let it be proposed to find the limiting value of its area, from $x=0$ to $x=\alpha$, on the supposition that p approaches continually to unity. First, we observe that

all the ordinates will be ultimately evanescent except those corresponding to $x=0$, 2π, 4π, &c., which will be very large, because for those values $\frac{1+p}{1-p}$ becomes a factor of the expression for y. We will begin by supposing that $a < 2\pi$; and therefore the only ordinates that we are concerned with are those immediately succeeding that through the origin; so that, making $p = 1-\rho$, and then supposing x very small, we get successively

$$y = \frac{\rho\,(2-\rho)\,e^{-kx}}{\rho^2 + 4\,(1-\rho)\sin^2\frac{1}{2}x} = \frac{2\rho}{\rho^2+x^2}, \text{ ultimately};$$

$$\therefore \int_0^a dxy = 2\tan^{-1}\frac{a}{\rho}, \text{ and } \int_0^a dxy_{p=1} = \pi, \text{ making } \rho = 0.$$

Next, suppose that the area is to be found from $x=0$ to $x=2\pi$; then, besides the area just found, there will be another portion immediately preceding the point for which $x=2\pi$; to find this latter portion put $x = 2\pi - x'$, and call the ordinate y';

$$\text{then } y' = \frac{(1-p^2)\,e^{-2k\pi+kx'}}{1-2p\cos x' + p^2} = e^{-2k\pi}\,y,$$

therefore the portion of the area immediately preceding the second limit $= \pi e^{-2k\pi}$;

$$\therefore \int_0^{2\pi} dxy_{p=1} = \pi + \pi e^{-2k\pi}.$$

120. As the following important theorems in Definite Integrals are based upon the preceding investigations, we shall here give the proofs of them. As in Art. 118 we have

$$y = \frac{(1-p^2)\,f(x)}{1-2p\cos(c-x)\,\frac{\pi}{a} + p^2} = f(x) + 2^1 S^\infty p^n \cos n\,(c-x)\,\frac{\pi}{a}\,.f(x);$$

$$\therefore \int_{-a}^{+a} dxy_{p=1} = \int_{-a}^{+a} dxf(x) + 2^1 S^\infty \int_{-a}^{+a} dx \cos n\,(c-x)\,\frac{\pi}{a}\,.f(x).$$

Now, suppose c to be less than a, then for all values of x between $x=-a$ and $x=+a$, y is evanescent, except when $x=c$; if therefore we write $x=c+z$, supposing z exceedingly small,

and then integrate with respect to z, from $z=0$ to $z=\gamma$ any small finite value, and double the result, and make $p=1$, we shall obtain the value of $\int_{-a}^{+a} dxy_{p=1}$. But making $p=1-\rho$, we have

$$y=\frac{\rho(2-\rho)f(c+z)}{\rho^2+4(1-\rho)\sin^2\frac{z\pi}{2a}}=\frac{2\rho f(c)}{\rho^2+\frac{\pi^2z^2}{a^2}} \text{ ultimately};$$

$$\therefore \int_0^\gamma dzy=\frac{2af(c)}{\pi}\tan^{-1}\frac{\pi\gamma}{\rho a}; \quad \int_0^\gamma dzy_{p=1}=af(c);$$

$$\therefore\ 2af(c)=\int_{-a}^{+a}dxf(x)+2^1S^\infty\int_{-a}^{+a}dx\cos n(c-x)\frac{\pi}{a}.f(x). \quad (1)$$

121. Hence

$$f(c)=\frac{1}{2a}\int_{-a}^{+a}dxf(x)+\frac{1}{a}\int_{a}^{+a}dx\left\{\cos(c-x)\frac{\pi}{a}.f(x)\right.$$

$$\left.+\cos(c-x)\frac{2\pi}{a}.f(x)+\cos(c-x)\frac{3\pi}{a}.f(x)+\ldots\right\};$$

therefore, making a infinite,

$$f(c)=\frac{1}{\pi}\text{ limit of }\frac{\pi}{a}\int_{-a}^{+a}dx\left\{\cos(c-x)\frac{\pi}{a}+\cos(c-x)\frac{2\pi}{a}+\text{\&c.}\right\}f(x),$$

or, $f(c)=\frac{1}{\pi}\int_0^\infty dz\left\{\int_{-a}^{+a}dx\cos(c-x)z\,.f(x)\right\}$ (Integ. Cal. Art. 116),

a theorem given by Fourier, and included, as we see, in the theorem (1), which was first given by Poisson.

122. When in physical questions a definite integral arises whose value cannot be exhibited in finite terms, or in a form convenient for numerical calculation, the method of Quadratures is used as a substitute for Integration. This method consists in taking a series of values of the function to be integrated, multiplying these by the differences of the corresponding values of the independent variable, and adding together all the results. The sum of such results approximates to the value of the de-

finite integral, as the intervals of the independent variable are diminished. The method is equivalent to adopting, for the area of a curve which a definite integral gives, the sum of the areas of a system of inscribed or circumscribed polygons.

Let the equal intervals into which the independent variable x is divided be taken equal to unity, and let $f(x)$ be the function of x to be integrated between assigned limits a and b. Take a value of it $f(a+n-\frac{1}{2})$ at the middle of the n^{th} of the intervals into which $b-a$ is divided. If the intervals be very small, $f(x)$ may be considered constant from

$$x = a + n - 1 \text{ to } x = a + n,$$

or $\Delta^2 f(x)$ may be neglected; and the value of the definite integral is $\Sigma f(a+n-\frac{1}{2})$, where n is to receive in succession the values

$$1,\ 2,\ 3, \ldots b-a.$$

For greater accuracy, suppose $\Delta^2 f(x)$ to be constant from

$$x = a \text{ to } x = b,$$

or $\Delta^3 f(x)$ to be neglected. Then

$$f(a+n-\tfrac{1}{2}+z) = f(a+n-\tfrac{1}{2}) + z\Delta f(a+n-\tfrac{1}{2}) + \frac{z^2}{2}\Delta^2 f(x),$$

and the integral for the interval in question is

$$\int dz f(a+n-\tfrac{1}{2}+z), \text{ from } z = -\tfrac{1}{2} \text{ to } z = \tfrac{1}{2},$$

$$= f(a+n-\tfrac{1}{2}) + \frac{1}{6}\left(\frac{1}{8}+\frac{1}{8}\right)\Delta^2 f(x) = f(a+n-\tfrac{1}{2}) + \frac{1}{24}\Delta^2 f(x).$$

The integral, consequently, from a to b, is

$$\Sigma\{f(a+n-\tfrac{1}{2}) + \frac{1}{24}\Delta^2 f(x)\};$$

n receiving the same successive values as in the former case. In calculating the value of a definite integral by this method, the intervals are to be taken smaller as the variation of the function is more rapid.

Convergency and Divergency of Series.

123. A series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ (to ∞) is called convergent, if the sum s_n of any number n of its terms approaches continually to a finite quantity s as its limit, when n is indefinitely increased; and divergent in the contrary case.

When a series is convergent, the sum of any number of consecutive terms after the n^{th} continually tends to zero, as n increases. For

$$u_{n+1} + u_{n+2} + \ldots + u_{n+m} = s_{n+m} - s_n;$$

therefore, as n increases, the value of the first member continually approaches to $s - s$, or zero, as its limit. This being true when $m = 1$, we see also that u_{n+1}, or the general term u_n continually tends to zero as n increases; that is, each term is greater than the following one; but this, although a necessary condition, is not sufficient to insure the convergency of a series. Thus in the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n};$$

$u_n = \frac{1}{n}$ continually approaches to zero as n increases; but

$$s_{n+m} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+m}$$

is evidently greater than $\frac{m}{n+m} > \frac{1}{2}$, if $m = n$; and therefore the sum of n consecutive terms after the n^{th}, does not diminish indefinitely as n increases; consequently the series in question is divergent.

124. In the geometrical progression

$$a + ax + ax^2 + \ldots + ax^{n-1} + \ldots \quad (1),$$

$$s_n = a \cdot \frac{1 - x^n}{1 - x}, \quad s_{n+m} = a \cdot \frac{1 - x^{n+m}}{1 - x};$$

$$\therefore\ s_{n+m} - s_n = ax^n \cdot \frac{1 - x^m}{1 - x}.$$

Hence if $x < 1$, when n is infinite $x^n = 0$,

$$\therefore\ s = \frac{a}{1-x}; \quad \text{also } u_n = 0,\ s_{n+m} - s_n = 0;$$

both which results shew that the series is convergent.

But if $x > 1$, then $u_n = ax^{n-1}$ increases indefinitely with n, which alone shews that the series is divergent.

Hence the geometrical progression (1) is convergent or divergent, according as x is less or greater than unity; and it may be used as the test of the convergency or divergency of other series. For if a proposed series can be shewn to have no term greater than the corresponding term of (1) when $x < 1$, then that series is convergent; or if a proposed series can be shewn to have no term less than the corresponding term of (1) when $x > 1$, then that series is divergent.

Thus in the series for e the base of the natural logarithms

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \dots + \frac{1}{\lfloor n-1} + \dots$$

the terms which follow the n^{th}, viz.

$$\frac{1}{\lfloor n} + \frac{1}{\lfloor n+1} + \frac{1}{\lfloor n+2} + \text{\&c.}$$

are evidently less than the corresponding terms of the geometrical progression

$$\frac{1}{\lfloor n} + \frac{1}{\lfloor n} \cdot \frac{1}{n} + \frac{1}{\lfloor n} \cdot \frac{1}{n^2} + \text{\&c.}$$

the sum of which is

$$\frac{1}{\lfloor n} \cdot \frac{1}{1 - \frac{1}{n}} = \frac{1}{\lfloor n-1} \cdot \frac{1}{n-1},$$

and continually tends to zero as n increases. Therefore the proposed series is convergent; and the error in taking the aggregate of its first n terms for its sum, is less than the quotient of the n^{th} term divided by $n - 1$.

125. From the measure of convergency or divergency which a geometrical progression furnishes, we shall now proceed to deduce one or two other tests as given by Cauchy (*Cours d'Analyse Algébrique*).

The series $u_1 + u_2 + \ldots + u_n + u_{n+1} \ldots$ is convergent if the limit of the ratio of u_{n+1} to u_n when n is infinite, be less than unity; and divergent in the contrary case. Suppose that as n increases indefinitely, the ratio $\frac{u_{n+1}}{u_n}$ continually tends to become equal to a finite quantity k as its limit; and let i denote a number less than the difference between k and 1, so that the quantities $k - i$ and $k + i$ are both less than 1 when k is less than 1, and both greater than 1 in the contrary case. Then by giving n a sufficiently large value the ratio $\frac{u_{n+1}}{u_n}$ may be made to lie between $k - i$ and $k + i$ for all superior values of n; and in the series

$$\begin{array}{llll} u_n + u_n (k-i) & + u_n (k-i)^2 & + \ldots \\ u_n + u_{n+1} & + u_{n+2} & + \ldots \\ u_n + u_n (k+i) & + u_n (k+i)^2 & + \ldots \end{array}$$

every term of the second will be intermediate in value to the corresponding terms of the first and third. But the first and third series evidently decrease indefinitely if u_n does so, as n is increased without limit, provided k be less than 1; therefore the second series which is intermediate in value to these is convergent under the same circumstances; and therefore the proposed series is convergent provided the limit of the ratio of u_{n+1} to u_n when n is infinite, be less than 1. Also the three series above written will all increase indefinitely if k be greater than 1, and the proposed series is in that case divergent. The test of convergency or divergency is here presented in the form most convenient for application; but it may be changed into the following: if $k = 1$, the test gives no result in either form.

126. The series $u_1 + u_2 + \ldots + u_n + \ldots$ is convergent, or will become so, if the superior limit of $(u_n)^{\frac{1}{n}}$ be less than 1, when n is infinite; and divergent in the contrary case.

Let k denote the superior limit of $(u_n)^{\frac{1}{n}}$ when n is infinite; and first suppose $k < 1$; also, let a be any magnitude between k and 1, so that $k < a < 1$; then when n is increased indefinitely, $(u_n)^{\frac{1}{n}}$ cannot approach indefinitely near to k without finally becoming constantly less than a. Therefore it will be possible to take for n so large a value, that for that and all superior values, we may constantly have

$$(u_n)^{\frac{1}{n}} < a, \quad u_n < a^n.$$

Consequently, the proposed series will finish by always having its terms less than the corresponding terms of the geometrical progression

$$a + a^2 + \dots + a^n + a^{n+1} + \dots;$$

and as this series is convergent, a being < 1, it follows that the proposed series will *à fortiori* end by being convergent.

Secondly, suppose $k > 1$; and take, as before, a between 1 and k, so that $k > a > 1$. Then, when n is indefinitely increased, $(u_n)^{\frac{1}{n}}$ cannot approach indefinitely near to k without finally becoming constantly $> a$; we shall therefore be able to satisfy the condition $(u_n)^{\frac{1}{n}} > a$, or $u_n > a^n$, by taking n sufficiently large; and consequently we shall always find in the series

$$u_1 + u_2 + \dots + u_n + u_{n+1} + \dots,$$

an indefinite number of terms greater than the corresponding terms of the geometrical progression

$$a + a^2 + \dots a^n + a^{n+1} + \dots,$$

which is divergent, a being > 1; and therefore the proposed series will end by being divergent.

127. It may be shewn that if as n increases indefinitely u_n remains positive, and the ratio $\frac{u_{n+1}}{u_n}$ continually tends to become equal to a finite quantity k as its limit, the expression $(u_n)^{\frac{1}{n}}$

continually approaches to the same limit. For as in Art. 125, by giving to m a value large enough, we may make each of the ratios $\frac{u_{m+1}}{u_m}, \frac{u_{m+2}}{u_{m+1}}, \ldots \frac{u_{m+n}}{u_{m+n-1}}$ differ from k by a quantity as small as we please; and consequently the geometric mean between these ratios $\left(\frac{u_{m+n}}{u_m}\right)^{\frac{1}{n}}$ may, by sufficiently increasing n, be made to differ from k by a quantity i as small as we please. We shall therefore have, taking n sufficiently large,

$$(u_{m+n})^{\frac{1}{n}} = \left\{u_{n\left(1+\frac{m}{n}\right)}\right\}^{\frac{1}{n}} = (k \pm i)\,(u_m)^{\frac{1}{n}}.$$

Now suppose n infinite, so that i becomes equal to zero, and we get

$$\text{limit of } (u_n)^{\frac{1}{n}}_{n=\infty} = k = \text{limit of } \left(\frac{u_{n+1}}{u_n}\right)_{n=\infty}.$$

128. Suppose that the series $u_1 + u_2 + u_3 + \ldots$ consists of both positive and negative terms; then if v_1, v_2, v_3, &c. be the numerical values of these terms, so that $u_1 = \pm v_1$, $u_2 = \pm v_2$, &c., it is evident that the sum of the proposed series can never surpass that of the series $v_1 + v_2 + v_3 + \ldots$; if therefore the latter series be convergent, that is, if $\left(\frac{v_{n+1}}{v_n}\right)_{n=\infty} < 1$, the proposed series will be convergent; or if the latter series finish by having terms greater than any assignable magnitude, that is, if $\left(\frac{v_{n+1}}{v_n}\right)_{n=\infty} > 1$, the same thing will happen to the proposed series, which will consequently be divergent. Hence the above test is applicable to series consisting of both positive and negative terms, provided we use the numerical values of the terms without regard to signs; and it fails, as in the preceding case, when $k = 1$.

129. Let $$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \quad (1)$$

be a series arranged according to positive and ascending powers of the variable x; the coefficients being positive or negative; then, by what has been proved, this series will be convergent or divergent, according as kx,

$$\text{where } k = (a_n)^{\frac{1}{n}} = \frac{a_{n+1}}{a_n} \text{ when } n \text{ is infinite,}$$

is numerically less or greater than 1. Hence for all values of x between $-\frac{1}{k}$ and $+\frac{1}{k}$ the series will be convergent, and for all values of x beyond those limits it will be divergent.

Ex. 1. $\frac{a}{1}x + \frac{a(a-1)}{1.2}x^2 + \frac{a(a-1)(a-2)}{1.2.3}x^3 + \&c.$

Here $\frac{a_{n+1}}{a_n} = \frac{a-n}{n+1} = -1$, when n is infinite;

therefore the series is convergent or divergent, according as x lies between $+1$ and -1, or without those limits.

Ex. 2. $$\frac{a}{1} + \frac{a^2}{2} + \frac{a^3}{3} + \&c.,$$

$$u_n = \frac{a^n}{n}, \quad \therefore \frac{u_{n+1}}{u_n} = \frac{na}{n+1} = a \text{ when } n \text{ is infinite;}$$

therefore the series is convergent or divergent, according as $a <$ or > 1.

Ex. 3. $$\frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots + \frac{1}{\underline{|n}},$$

$$\frac{u_{n+1}}{u_n} = \frac{1}{n+1} = 0, \text{ when } n \text{ is infinite,}$$

therefore the series converges. Similarly, it will appear that the series for $\sin x$ and $\cos x$ are always convergent; and that for $\tan^{-1} x$ convergent for all values of x situated between $+1$ and -1.

130. The series $u_1 + u_2 + \ldots + u_n + \ldots$ is convergent, or will become so, if the inferior limit of $\frac{\log u_n}{\log \frac{1}{n}}$ be > 1, when n is infinite; and divergent in the contrary case.

Let k denote the inferior limit of $\dfrac{\log u_n}{\log \dfrac{1}{n}}$ when n is infinite, and first suppose $k > 1$; also let a be any number between k and 1, so that $k > a > 1$. Then when n is indefinitely increased, $\dfrac{\log u_n}{\log \dfrac{1}{n}}$ or its equal $\dfrac{\log \dfrac{1}{u_n}}{\log n}$ cannot approach indefinitely near to k without finally becoming constantly greater than a. Therefore it will be possible to take for n so large a value, that for that and all superior values we may constantly have

$$\log \frac{1}{u_n} > a \log n > \log n^a, \quad \text{or } \frac{1}{u_n} > n^a, \quad \text{or } u_n < \frac{1}{n^a}.$$

Consequently the proposed series will finish by always having its terms less than the corresponding terms of the converging series (Ex. Art. 131)

$$\frac{1}{1^a} + \frac{1}{2^a} + \frac{1}{3^a} + \dots + \frac{1}{n^a} + \frac{1}{(n+1)^a} + \&c.$$

and therefore will itself end by being, *à fortiori*, convergent.

Similarly, if $k < 1$, it may be shewn that the proposed series will finish by being divergent.

131. The series $u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \&c. \dots\dots\dots (1)$, each term of which is less than the immediately preceding term,

and the series $u_0 + 2u_1 + 4u_3 + 8u_7 + 16u_{15} + \&c. \dots\dots (2)$,

are convergent or divergent at the same time.

Suppose series (1) to be convergent and that its sum $= s$,

$$\text{then } u_0 = u_0, \quad 2u_1 = 2u_1, \quad 4u_3 < 2u_2 + 2u_3,$$

$$8u_7 < 2u_4 + 2u_5 + 2u_6 + 2u_7, \ \&c.;$$

$$\therefore\ u_0 + 2u_1 + 4u_3 + 8u_7 + 16u_{15} + \&c.$$

$$< u_0 + 2(u_1 + u_2 + u_3 + \&c.) < 2s - u_0,$$

consequently series (2) is convergent. Next, suppose series (1) divergent, then

$$u_0 = u_0, \quad 2u_1 > u_1 + u_2, \quad 4u_3 > u_3 + u_4 + u_5 + u_6, \text{ \&c.};$$

$$\therefore\ u_0 + 2u_1 + 4u_3 + 8u_7 + \text{\&c.} > u_0 + u_1 + u_2 + u_3 + \text{\&c.},$$

and is therefore divergent.

Ex. Let series (1) be $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \text{\&c.} \ldots\ldots (3)$,

then series (2) is $\frac{1}{1^m} + \frac{1}{2^{m-1}} + \frac{1}{4^{m-1}} + \frac{1}{8^{m-1}} + \text{\&c.}$,

a geometric progression convergent when $m > 1$, and divergent in the contrary case; consequently series (3) will be convergent if $m > 1$, and divergent if $m =$ or < 1.

132. The alternating series $u_1 - u_2 + u_3 - u_4 + \text{\&c.}$, is convergent, if the numerical value of the terms decreases without limit.

For, by writing it in the forms

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - \text{\&c.},$$

$$u_1 - u_2 + (u_3 - u_4) + \text{\&c.};$$

we see that it is $> u_1 - u_2$ and $< u_1$, and therefore is convergent.

Thus the sum of the series $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{\&c.}$,

lies between 1 and $\frac{1}{2}$; also the series $\frac{1}{1^m} - \frac{1}{2^m} + \frac{1}{3^m} - \frac{1}{4^m} + \text{\&c.}$,

is convergent for all positive values of m.

133. It is also shewn in the Work from which these tests of convergency are taken, that two converging series all whose terms are positive, will by their addition or multiplication produce new converging series, whose sums result from the addition or multiplication of the sums of the former. If the proposed

series be imaginary, having its general term of the form $v_n + w_n\sqrt{-1}$, it may be resolved into the two series

$$v_1 + v_2 + \ldots + v_n + \ldots, \quad \sqrt{-1}\,(w_1 + w_2 + \ldots + w_n + \ldots) \ldots (1),$$

to which the foregoing tests may be applied separately. If we assume

$$v_n + w_n\sqrt{-1} = \rho_n(\cos\theta_n + \sqrt{-1}\,\sin\theta_n),$$

then $v^2_n + w^2_n = \rho^2_n$, so that v_n and w_n are both less than ρ_n; if therefore the series of moduli $\rho_1 + \rho_2 + \ldots + \rho_n + \ldots$ be convergent, then each of the series (1) will be convergent.

Interpolation of Series.

134. When a series of values of a quantity has been obtained either by observation or calculation, it is of great importance to be able to insert other values between them, such as would have resulted from a similar observation or calculation, without the labour of performing these. This is the object of Interpolation; and in this the Calculus of Finite Differences finds one of its chief uses. More strictly, Interpolation of Series is the inserting among the terms of a given series, new terms subject to the same law as the first. In doing this, the terms of the series are considered as particular values of the function which expresses its general term, corresponding to a given regular succession of indices; and it is the business of Interpolation to discover that general term; or at least to assign such a function of the index as shall represent the given series of values, and, approximately, all intermediate values. The problem thus requiring us to assign the analytical expression of a function from a limited number of its numerical values, is plainly indeterminate; it is the same as to form the equation to a curve which shall pass through a limited number of points, whose abscissæ represent the values of the independent variable, and the ordinates those of the function, without giving the species of the curve; which, as is evident, may be done in an infinite variety of ways. But if the given terms are numerous and near to each other, the expression

for the general term, within the limits of the given quantities, may be found to a great degree of accuracy.

135. There are two principal cases to be considered; first, when the given values of $f(x)$, namely,

$$f(x_1),\quad f(x_1+h),\quad f(x_1+2h),\ldots\quad f\{x_1+(n-1)\,h\},$$

$$\text{or } u_1,\quad u_2,\quad u_3,\ldots\quad u_n,$$

as we shall write them, correspond to values of the independent variable, x_1, x_1+h, x_1+2h, &c., $x_1+(n-1)\,h$, in arithmetical progression. And secondly, when the given values $f(x_1)$, $f(x_2)$, $\ldots f(x_n)$, or u_1, $u_2,\ldots u_n$, correspond to values x_1, $x_2,\ldots x_n$, of the independent variable, not obeying any assigned law.

136. Having given u_1, u_2, u_3, &c. u_n, n values of a function $f(x)$, corresponding to the n values of the independent variable x_1, $x_1+h,\ldots x_1+(n-1)\,h$, to find an expression for any intermediate value $f(x_1+k)$.

Since $f(x+nh)=f(x)+n\Delta f(x)+\dfrac{n(n-1)}{1\,.\,2}\Delta^2 f(x)+\text{\&c.}$

(Art. 23), changing x into x_1, and then replacing nh by k, and n by $\dfrac{k}{h}$ we get

$$f(x_1+k)=u_1+\frac{k}{h}\Delta u_1+\frac{k(k-h)}{1\,.\,2h^2}\Delta^2 u_1+\text{\&c.}$$

$$+\frac{k(k-h)\ldots\{k-(n-2)\,h\}}{\underline{\lfloor n-1}\,.\,h^{n-1}}\Delta^{n-1}u_1;$$

in which, since

$$\left.\begin{aligned}\Delta u_1\ &=u_2-u_1,\\ \Delta^2 u_1\ &=u_3-2u_2+u_1,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ \Delta^{n-1}u_1&=u_n-(n-1)\,u_{n-1}+\frac{(n-1)(n-2)}{1\,.\,2}u_{n-2}-\text{\&c.}\ldots\pm u_1,\end{aligned}\right\}\ (2)$$

if we make $k=0$, h, $2h$, &c., $(n-1)\,h$, the second member assumes the n values u_1, $u_2,\ldots u_n$; and not only this, but if we assume for k any value whatever between 0 and $(n-1)\,h$, we

shall obtain the value of the function corresponding to that value of the independent variable.

137. In applying the above formula, the simplest mode is not to calculate Δu_1, $\Delta^2 u_1$, &c., by equations (2), but by continued subtraction of the given terms; that is, we must write down the series of given values $u_1, u_2, \ldots u_n$, and subtract each from the succeeding one; next subtract each of these differences from the succeeding difference; then perform the same operation upon the new differences; and so on, till the process terminates; the first terms of these series of differences are the values of $\Delta u_1, \Delta^2 u_1, \ldots \Delta^{n-1} u_1$. Unless the terms of the given series by continued subtraction lead to a constant difference, the expression for $f(x_1 + k)$ will have as many terms as the given series of values has.

Ex. 1. Having given the values of sin 30°, sin 31°, sin 32°, sin 33°, to find $\sin(30^\circ + k')$, k being between 0 and 180′. Here

$u_1 = .5$	Δ		
$u_2 = .5150381$	150381	Δ^2	
$u_3 = .5299193$	148812	-1569	Δ^3
$u_4 = .5446390$	147197	-1615	-46;

$$\therefore\ \Delta u_1 = 0.0150381,\ \Delta^2 u_1 = -0.0001569,\ \Delta^3 u_1 = -0.0000046,$$

$$\therefore\ \sin(30^\circ + k') = .5 + \frac{k}{60}\Delta u_1 + \frac{k(k-60)}{2 \cdot 60^2}\Delta^2 u_1$$

$$+ \frac{k(k-60)(k-120)}{2 \cdot 3 \cdot 60^3}\Delta^3 u_1.$$

If $k = 20$, it will be found that $\sin 30^\circ.\,20' = .5050299$, which is too large only by a unit in the seventh place of decimals.

Ex. 2. Having given

$$\log 3.14 = 0.496929,\quad \log 3.15 = 0.498310,$$

$$\log 3.16 = 0.499687,\quad \log 3.17 = 0.501059;$$

shew that $\log 3.14159 = 0.497149$.

Ex. 3. Having given u_1, u_2, u_3, u_4, four right ascensions (declinations, longitudes, &c.) of the Moon at intervals of 12 hours, to find its value t hours after the time corresponding to the second value. Here $h = 12$, $k = 12 + t$, and if the required right ascension $= u_2 + \delta$, then

$$u_2 + \delta = u_1 + \left(1 + \frac{t}{12}\right) \Delta u_1 + \frac{(t+12)\,t}{2\,.\,12^2} \Delta^2 u_1 + \frac{(t+12)\,t\,(t-12)}{2\,.\,3\,.\,12^3} \Delta^3 u_1,$$

$$\text{or } \delta = \frac{t}{12}\left(\Delta^1 + \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3\right) + \frac{1}{2}\left(\frac{t}{12}\right)^2 \Delta^2 + \frac{1}{6}\left(\frac{t}{12}\right)^3 \Delta^3,$$

when developed in powers of $\frac{t}{12}$, Δ^1 being written for Δu_1, Δ^2 for $\Delta^2 u_1$, &c.

138. Between every two consecutive terms of a given series, to interpolate any number of equidistant terms.

Let u_1, u_2, u_3, &c., u_n be the given series, and let $m - 1$ be the number of equidistant terms to be inserted between every two consecutive terms; then the new series will be

$$u_1,\quad u_{\frac{m+1}{m}},\quad u_{\frac{m+2}{m}}, \ldots\ u_{\frac{2m-1}{m}},\quad u_2,\quad u_{\frac{2m+1}{m}}, \text{ \&c.}$$

If therefore v_{r+1} denote the $r+1^{\text{th}}$ term of this series, we have

$$v_{r+1} = u_{\frac{m+r}{m}} = f\left(1 + \frac{r}{m}\right),$$

$$\text{or } v_{r+1} = u_1 + \frac{r}{m}\Delta u_1 + \frac{r\,(r-m)}{1\,.\,2m^2}\Delta^2 u_1 + \text{\&c.}$$

Hence, taking r from 1 to $m - 1$, we get the terms inserted between u_1 and u_2; next taking r from $m + 1$ to $2m - 1$, we get the terms between u_2 and u_3; and so on. The differences Δu_1, $\Delta^2 u_1$, &c., are to be computed by continued subtraction as in Art. (137); and the series for v_{r+1} will have as many terms as the proposed series has, unless those terms by continued subtraction lead to a constant difference.

Ex. To insert three equidistant terms between every two

consecutive ones of the series 1, 7, 15, 28, 49, &c. Here $m = 4$, and

$$\begin{array}{lllll} \Delta, & 6, & 8, & 13, & 21, \\ \Delta^2 & & 2, & 5, & 8, \\ \Delta^3 & & & 3, & 3, \end{array}$$

$$\therefore\ v_{r+1} = 1 + \frac{6r}{4} + \frac{r(r-4)}{16} + \frac{r(r-4)(r-8)}{128} = \frac{128 + 192r - 4r^2 + r^3}{128};$$

and the series is $1,\ \frac{317}{128},\ \frac{504}{128},\ \frac{695}{128},\ 7,\ \frac{1113}{128}$, &c.

139. The formula of Art. 136 may be presented under a different form by changing k into $x - x_1$, which gives

$$f(x) = u_1 + \frac{x - x_1}{h}\Delta u_1 + \frac{(x - x_1)(x - x_1 - h)}{1.2.h^2}\Delta^2 u_1 + \&c.$$

$$+ \frac{(x - x_1)(x - x_1 - h)\ldots\{x - x_1 - (n-2)h\}}{\lfloor n-1 . h^{n-1}}\Delta^{n-1}u_1;$$

where $f(x)$ is a function of x which, as x assumes the n values x_1, $x_1 + h$, &c., $x_1 + (n-1)h$, successively assumes the corresponding values $u_1, u_2, \ldots u_n$; and for any other value of x within, or not far beyond, the limits x_1 and $x_1 + (n-1)h$, it gives the value of the corresponding interpolated term. If we put $h = 1$, the formula is adapted to the case where the increment of the principal variable is unity.

140. In any series of consecutive equidistant values of a function, where one is deficient to insert that one.

Let u_1, u_2, u_3, &c., u_n be the values of the function corresponding to the values $x_1, x_1 + h, \ldots x_1 + (n-1)h$ for x. Then assuming that $\Delta^{n-1}u_1 = 0$, or that the $(n-2)^{\text{th}}$ differences are constant, which will almost always be the case in tabulated results, we have

$$\Delta^{n-1}u_1 = u_n - (n-1)u_{n-1} + \frac{(n-1)(n-2)}{1.2}u_{n-2} - \&c. \pm u_1 = 0,$$

an equation of the first degree from which any one of the values as u_r may be found, if the rest be known. Having thus completed the system of values, we may interpolate any intermediate term $f(x_1+k)$ by the method of Art. 136. If two values out of n are deficient, then we must suppose

$$\Delta^{n-2}u_1=0, \quad \Delta^{n-2}u_2=0, \text{ or}$$

$$u_{n-1}-(n-2)\,u_{n-2}+\frac{(n-2)(n-3)}{1\,.\,2}\,u_{n-3}-\&\text{c.} \pm u_1=0,$$

$$u_n-(n-2)\,u_{n-1}+\frac{(n-2)(n-3)}{1\,.\,2}\,u_{n-2}-\&\text{c.} \pm u_2=0\,;$$

which equations will suffice to determine any two of the values in terms of the rest. In the same way any number of deficient terms may be inserted.

Ex. 1. Given the cube roots of 121, 122, 124, 125, to find that of 123.

$$u_1=4.946088, \quad u_2=4.959675, \quad u_4=4.986631. \quad u_5=5,$$

$$\Delta^4 u_1=u_5-4u_4+6u_3-4u_2+u_1=0\,;$$

$$\therefore\ u_3=\frac{1}{6}\{4\,(u_2+u_4)-u_1-u_5\}=4.973190.$$

Ex. 2. Given u_1, u_2, u_5, u_6, to find u_3, u_4,

$$\Delta^4 u_1=u_5-4u_4+6u_3-4u_2+u_1=0.$$

$$\Delta^4 u_2=u_6-4u_5+6u_4-4u_3+u_2=0\,;$$

$$\therefore\ u_3=\frac{1}{10}\,(-3u_1+10u_2+5u_5-2u_6),$$

$$u_4=\frac{1}{10}\,(-2u_1+5u_2+10u_5-3u_6).$$

Ex. 3. Given the logarithms of 121, 122, 125, 126 equal respectively to 2.0827854, 2.0863598, 2.0969100, 2.1003705; shew that the logarithms of 123, 124, are equal respectively to 2.0899051, 2.0934217.

141. We next come to the case where the given values

$$f(x_1), \quad f(x_2), \ \&\text{c.}, \quad f(x_n), \quad \text{or } u_1, \quad u_2, \ldots, u_n,$$

correspond to values $x_1, x_2, \ldots x_n$, not obeying any assigned law; and it is required to determine a rational integral function of $n-1$ dimensions, $f(x)$, which shall assume the n given values $u_1, u_2, \ldots u_n$, when for x the values x_1, x_2, x_3, &c., x_n are successively substituted.

Since $f(x)$ is of $(n-1)$ dimensions, we may assume

$$\frac{f(x)}{(x-x_1)(x-x_2)\ldots(x-x_n)} = \frac{C_1}{x-x_1} + \frac{C_2}{x-x_2} + \ldots + \frac{C_n}{x-x_n};$$

$$\therefore f(x) = C_1(x-x_2)(x-x_3)\ldots(x-x_n)$$

$$+C_2(x-x_1)(x-x_3)\ldots(x-x_n) + \&c. + C_n(x-x_1)(x-x_2)\ldots(x-x_{n-1}).$$

Now make $x = x_1, x_2$, &c., x_n, successively; and observing that the corresponding values of the first member are $u_1, u_2, \ldots u_n$, we get

$$u_1 = C_1(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n),$$

$$u_2 = C_2(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_n = C_n(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1});$$

$$\therefore f(x) = u_1 \frac{(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)}$$

$$+ u_2 \frac{(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} + \&c.$$

$$+ u_n \frac{(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})},$$

which is Lagrange's Theorem for interpolation.

Ex. To find a function of x which, when $x = 1, 3, 6, 12$, shall assume the values $1, 7, 10, -8$.

$$f(x) = -\frac{(x-3)(x-6)(x-12)}{2.5.11} + 7.\frac{(x-1)(x-6)(x-12)}{2.3.9}$$

$$-\frac{(x-1)(x-3)(x-12)}{3.3} - 4.\frac{(x-1)(x-3)(x-6)}{11.9.3}.$$

142. To determine the maximum or minimum value of a function, from three of its values near its maximum or minimum, and the three corresponding values of the independent variable.

If u_1, u_2, u_3 be the given values of u, and x_1, x_2, x_3 those of x, we have

$$u = u_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + u_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + u_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}.$$

Hence, putting $\frac{du}{dx} = 0$, we find

$$u_1 (x_2 - x_3)(2x - x_2 - x_3) + u_2 (x_3 - x_1)(2x - x_1 - x_3)$$
$$+ u_3 (x_1 - x_2)(2x - x_1 - x_2) = 0;$$

$$\therefore\ x = \frac{u_1 (x_2^2 - x_3^2) - u_2 (x_1^2 - x_3^2) + u_3 (x_1^2 - x_2^2)}{2u_1 (x_2 - x_3) - 2u_2 (x_1 - x_3) + 2u_3 (x_1 - x_2)},$$

the value of x at which u is a maximum or minimum. This formula is useful in various Astronomical problems, as for instance, to determine the meridian altitude of a heavenly body, when an observation exactly on the meridian cannot be obtained.

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